

New solutions to the $sl_q(2)$ -invariant Yang-Baxter equations at roots of unity

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We find new solutions to the Yang-Baxter equations with the R -matrices possessing $sl_q(2)$ symmetry at roots of unity, using indecomposable representations. The corresponding quantum one-dimensional chain models, which can be treated as extensions of the XXZ model at roots of unity, are investigated. We consider the case $q^4 = 1$. The Hamiltonian operators of these models as a rule appear to be non-Hermitian. Taking into account the correspondence between the representations of the quantum algebra $sl_q(2)$ and the quantum super-algebra $osp_t(1|2)$, the presented analysis can be extended to the latter case for the appropriate values of the deformation parameter.

1 Introduction

The solutions to the Yang-Baxter equations (YBE) for the quantum algebra $sl_q(2)$ [1, 2] when deformation parameter q is given by a root of unity [3, 4, 5] are widely investigated for irreducible ("spin", (semi-)cyclic and nilpotent) representations [6, 7]. In this work we would like to fill up the existing gap by considering indecomposable ones [4, 5, 8]. We show that use of these representations provides a large number of new solutions to the YBE and correspondingly a rich variety of the $sl_q(2)$ -invariant integrable models at roots of unity.

The solutions to the YBE with the given symmetry admit linear decomposition over the symmetry-invariant objects - projectors [9, 10]. Our strategy in looking for a new solution to the Yang-Baxter equations is straightforward. After substitution of the most general linear combination of the appropriate $sl_q(2)$ -invariant objects (projectors) into the YB equations, the latter ones are reduced to the set of the functional equations defined on the corresponding coefficients. At roots of unity it takes place a degeneration of the standard fusion rules of the quantum algebras, and it introduces some modifications in the formulation of the $R_{A'A''}$ -matrices, defined on the tensor product of two spaces, $A' \otimes A''$, in terms of the projectors. In this paper we consider the highest and lowest weight representations of the quantum algebra when q is a root of unity, and the analysis is restricted to the representations, which have their analogues at general q or are emerging from their fusions (so-called A -type representations [3, 4, 5]). They are grouped into two classes: irreducible spin-representations V (spin-irrep) and corresponding indecomposable representations \mathcal{I} . So the task is to define the structure of the R_{VV} -, $R_{V\mathcal{I}}$ - and $R_{\mathcal{I}\mathcal{I}}$ -matrices in terms of the projection operators, obtaining preliminarily all the variety of the projectors. At roots of unity the number of the projectors acting on the spaces of the tensor products $\mathcal{I} \otimes V$ or $\mathcal{I}' \otimes \mathcal{I}''$ becomes larger than the number of the projectors in the case of general q (when instead of \mathcal{I} a direct sum of two irreps stands), and it leads to the increasing of the number of the solutions to the YBE. The obtained solutions allow us to construct new integrable models with Hamiltonian operators invariant with respect to the mentioned quantum algebra at roots of unity. New solutions are found in this paper, particularly, for the case $q^2 = -1$. By means of them quantum integrable chain models are constructed with the fundamental spin-1/2 representations on the sites, using the fact, that four-dimensional indecomposable representation is a direct product of two spin-1/2 irreps.

Investigation of the solutions to the YBE using the B -type representations (including cyclic,

semi-cyclic and nilpotent irreps and corresponding indecomposable representations), which have no their analogues at general q [3, 4, 5], will be done afterwards.

A similar analysis would be valid also for the case of the quantum super-algebra $osp_t(1|2)$ [11, 12, 13, 14, 16], due to the existing correspondence between the representations of the quantum algebras $sl_q(2)$ and $osp_t(1|2)$ with $q = it^{1/2}$ [15, 14, 8, 17]. Note, however, that when $q = \pm i$ ($t = 1$) the mentioned correspondence does not take place, because the non-deformed super-algebra $osp(1|2)$ has no even-dimensional representations.

The paper is organized as follows: in the first section we review the known ways to find solutions to the YBE. The second and third sections are devoted correspondingly to the description of the new solutions found for the exceptional values of the deformation parameter q and to the construction of the corresponding integrable chain models. The YB equations at this case have a huge number of the solutions. We discuss three large classes of the solutions in Section 2. In Section 3 we consider some of the Hamiltonian operators corresponding to the obtained R -matrices chosen as (symmetric) representatives of each class of the solutions, displaying the variety of the resulting 1d quantum chain models. The fourth section briefly depicts the character of the dynamics of the systems possessing non-Hermitian and non-diagonalizable Hamiltonian operators which met in the third section. In the Appendix the projection operators are described in general terms (an addition to Sections 1.2 and 1.3), and for $q = i$, particularly.

1.1 $sl_q(2)$ algebra and Jimbo's relations for composite R -matrices.

We define the algebra relations and co-product for quantum algebra $sl_q(2)$ as

$$[e, f] = \frac{k-k^{-1}}{q-q^{-1}}, \quad q^2 ek = ke, \quad fk = q^2 kf, \quad (1.1)$$

$$\Delta[e] = e \otimes k^{-1/2} + k^{1/2} \otimes e, \quad \Delta[f] = f \otimes k^{-1/2} + k^{1/2} \otimes f, \quad \Delta[k] = k \otimes k, \quad (1.2)$$

$$R\Delta = \bar{\Delta}R. \quad (1.3)$$

Here R is an intertwiner matrix characteristic to the quasi-triangular Hopf algebra, and $\bar{\Delta} = P\Delta P$, where P is a permutation operator $P : A' \otimes A'' = A'' \otimes A'$. The co-product Δ is a co-associative operation: $\Delta(1 \otimes \Delta) = \Delta(\Delta \otimes 1)$. The intertwiner matrix R satisfies to the constant Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (1.4)$$

R_{ij} acts on the tensor product of two representation spaces of the algebra, $A_i \otimes A_j$. Irreducible representations of $sl_q(2)$ at general q are classified similar to the spin-irreps of the non-deformed algebra $sl(2)$: r -dimensional irrep V_r is characterized by the spin value $j = (r-1)/2$. The quadratic Casimir operator, defined as

$$c = fe + (qk + q^{-1}k^{-1})/(q - q^{-1})^2, \quad (1.5)$$

has the eigenvalue $[r/2]_q^2 + \frac{2}{(q - q^{-1})^2}$ on V_r . The tensor product of two irreps has linear decomposition,

$$V_{r_1} \otimes V_{r_2} = \bigoplus_{r=|r_2-r_1|+1}^{r_2+r_1-1} V_r, \quad \Delta r = 2. \quad (1.6)$$

In this paper we denote the Casimir operator c acting on the space $V_{r_1} \otimes V_{r_2} \otimes \cdots V_{r_p}$ also as $c^{r_1 r_2 \cdots r_p}$.

In the theory of the integrable models the solutions $R_{ij}(u)$ to the Yang-Baxter equations with spectral parameter [21],

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v), \quad (1.7)$$

acquire an important role. The solutions of (1.7) are defined up to the following multiplicative transformations: $R_{ij}(u) \rightarrow f(u)R_{ij}(au)$, with arbitrary number a and arbitrary function $f(u)$. Jimbo's construction gives an opportunity to derive solutions to (1.7) from algebraic relations [10, 12, 13]. In the work [10] the author stated that Eq. (1.7) must be satisfied, if the matrix $R_{ij}(u)$ obeys the relations

$$\begin{aligned} \check{R}(u) (q^u f \otimes k^{1/2} + q^{-u} k^{-1/2} \otimes f) &= \\ &= (q^{-u} f \otimes k^{1/2} + q^u k^{-1/2} \otimes f) \check{R}(u), \\ \check{R}(u) (q^u k^{-1/2} \otimes e + q^{-u} e \otimes k^{1/2}) &= \\ &= (q^{-u} k^{-1/2} \otimes e + q^u e \otimes k^{1/2}) \check{R}(u). \end{aligned} \quad (1.8)$$

Here $\check{R}(u) = PR(u)$, for which

$$[\check{R}(u), \Delta] = 0. \quad (1.9)$$

When $q^n = 1$ [3, 4, 5, 22], then the number of the permissible irreducible representations is restricted: the irreps V_r can be of dimensions $r = 1, \dots, \mathcal{N}$, where $\mathcal{N} = n$, if n is odd and

$\mathcal{N} = n/2$, if n is even. The center of the algebra is enlarged, new Casimir operators appear, which are $e^{\mathcal{N}}$, $f^{\mathcal{N}}$ and $k^{\mathcal{N}}$. The irreducible representations are grouped into two types: A -type, which includes ordinary spin representations ($e^{\mathcal{N}} = 0$, $f^{\mathcal{N}} = 0$ and $k^{\mathcal{N}} = \pm 1$) with dimensions $\leq \mathcal{N}$, and B -type, which consists of cyclic ($e^{\mathcal{N}} \neq 0$, $f^{\mathcal{N}} \neq 0$), semi-cyclic ($e^{\mathcal{N}} \neq 0$, $f^{\mathcal{N}} = 0$ or $e^{\mathcal{N}} = 0$, $f^{\mathcal{N}} \neq 0$) and nilpotent representations ($e^{\mathcal{N}} = 0$, $f^{\mathcal{N}} = 0$ and $k^{\mathcal{N}} \neq \pm 1$) with dimensions equal to \mathcal{N} .

Among the non-reducible representations of the quantum algebra together with the irreducible representations there are also indecomposable ones, $\mathcal{I}_{A/B}$, of dimension $\mathcal{R} = 2\mathcal{N}$ [3, 4, 5, 8, 13, 16, 17]. It is known that A -type representations are forming a closed fusion ring [3, 4, 8]. We borrow from the work [8] the notations for A -type indecomposable representations, $\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}$, where r ($r > \mathcal{N}$) is the dimension of the maximal proper subspace of $\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}$, denoted below by an abstract notation \mathcal{U} : it has $(\mathcal{R} - r)$ -dimensional proper irreducible subspace U . In the fusions indecomposable representation $\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}$ arises from the "merging" of the representations V_r and $V_{\mathcal{R}-r}$ at roots of unity, when $c_r = c_{\mathcal{R}-r}$ and $V_r \Rightarrow \mathcal{U}$, $V_{\mathcal{R}-r} \Rightarrow U$ (see for details [3, 4, 8]).

We have excluded from the present consideration the highest/lowest weight nilpotent representations, for which $k^{\mathcal{N}}$ is generic. But of course, such kind of investigation, which is done in this work, can be carried out for them as well, previously making proper changes in the definitions of the indecomposable representations, as the representations in this case are parameterized by a continuous parameter (the value of $k^{\mathcal{N}}$). Also all the representations of B -type can be considered. As it is known the fusion of the B -type representations can contain the indecomposable representations of A -type [4]. Therefore the investigation of the solutions to the YBE for the representations of B -type will include the results of this paper particularly. All these questions we are addressing to our subsequent investigations.

In order to write down equations for indecomposable representations, similar to Eqs. (1.8), which lead to a simpler set of algebraic equations instead of the functional ones, let us write the Yang-Baxter equations with Lax operator L [10] (below r_i denotes the dimension of the representation, on which the operator acts):

$$R^{r_1 r_2}(u - v) L^{r_1}(u) L^{r_2}(v) = L^{r_2}(v) L^{r_1}(u) R^{r_1 r_2}(u - v), \quad (1.10)$$

where L^r is 2×2 matrix with operator-valued elements acting on the space V_r

$$L^r(u) = q^u L_+ - q^{-u} L_- , \quad L_+ = \begin{pmatrix} k^{1/2} & g_f f \\ 0 & k^{-1/2} \end{pmatrix}, \quad L_- = \begin{pmatrix} k^{-1/2} & 0 \\ g_e e & k^{1/2} \end{pmatrix}. \quad (1.11)$$

We take $g_f = \frac{q^2-1}{q^{3/2}}$ and $g_e = \frac{1-q^2}{q^{1/2}}$. The relations (1.8) can be obtained from the equation (1.10), expanding r.h.s. and l.h.s. of the latter in powers of q^v and taking the expressions linear in respect of q^v (or q^{-v}). In the case, when one of the representations, on which R_{12} acts, say the second one, is a composite one (i.e. can be represented as $V_{r'_2} \otimes V_{r''_2}$), then L^{r_2} must be modified. A natural generalization is to replace the algebra generators e, f, k in the expression (1.11) of L^{r_2} by the co-products $\Delta[e], \Delta[f], \Delta[k]$. It will give $\check{R}^{r_1 \ r'_2 \times r''_2}$ -matrix, which after multiplication from the left and right sides by proper projectors $1 \otimes P^r$ ($P^r \otimes 1$), becomes $R^{r_1 \ r}$, where $(|r_1 - r_2| + 1) \leq r \leq (r_1 + r_2 - 1)$. We do not consider the possibility of $(P^{r'} \otimes 1) \check{R}^{r_1 \ r'_2 \times r''_2} (1 \otimes P^{r''})$, with $r' \neq r''$, as the \check{R} -matrices are defined so that they are commuting with the algebra generators (1.9).

If we want to take into account the entire space of the fusion representations, we can write down $L^{r'_2 \times r''_2}$ as the following tensor product $L^{r'_2}(u) \otimes L^{r''_2}(w)$.

$$\check{R}^{r_1 \ r'_2 \times r''_2}(u-v, u-w) L^{r_1}(u) \left[L^{r'_2}(v) \otimes L^{r''_2}(w) \right] = \left[L^{r'_2}(v) \otimes L^{r''_2}(w) \right] L^{r_1}(u) \check{R}^{r_1 \ r'_2 \times r''_2}(u-v, u-w). \quad (1.12)$$

Besides of the usual commutativity relations $\check{R}^{r_1 \ r'_2 \times r''_2} \Delta(\Delta[a]) = \Delta(\Delta[a]) \check{R}^{r_1 \ r'_2 \times r''_2}$, $a = e, f, k^\pm$, the non-diagonal elements of the matrix-relations (1.12) contain also spectral parameter dependent relations, which are more complicated than (1.8): we shall refer to them as Jimbo's relations for composite (including tensor products of the irreps) representations. Here we write the following equations for the generator f (we suppose $v = w$ in (1.12), and $\check{R}^{r_1 \ r'_2 \times r''_2}(u, u) \equiv \check{R}(u)$)

$$\begin{aligned} & \check{R}(u) \left(q^u (\Delta[f] \otimes k^{\frac{1}{2}} + k^{\frac{1}{2}} \otimes k^{-\frac{1}{2}} \otimes f - \frac{(1-q^2)^2}{q^2} f \otimes e \otimes f + f \otimes k^{\frac{1}{2}} \otimes k^{-\frac{1}{2}}) + q^{-u} k^{-\frac{1}{2}} \otimes \Delta[f] \right) \\ &= \left(q^{-u} \Delta[f] \otimes k^{\frac{1}{2}} + q^u (k^{-\frac{1}{2}} \otimes \Delta[f] + k^{\frac{1}{2}} \otimes k^{-\frac{1}{2}} \otimes f - \frac{(1-q^2)^2}{q^2} f \otimes e \otimes f + f \otimes k^{\frac{1}{2}} \otimes k^{-\frac{1}{2}}) \right) \check{R}(u). \end{aligned} \quad (1.13)$$

and

$$\begin{aligned} & \check{R}(u) \left(q^u f \otimes k^{\frac{1}{2}} \otimes k^{\frac{1}{2}} + q^{-u} (k^{-\frac{1}{2}} \otimes f \otimes k^{\frac{1}{2}} + k^{-\frac{1}{2}} \otimes k^{-\frac{1}{2}} \otimes f) \right) \\ &= \left(q^u k^{-\frac{1}{2}} \otimes k^{-\frac{1}{2}} \otimes f + q^{-u} (k^{-\frac{1}{2}} \otimes f \otimes k^{\frac{1}{2}} + f \otimes k^{\frac{1}{2}} \otimes k^{\frac{1}{2}}) \right) \check{R}(u). \end{aligned} \quad (1.14)$$

In case of $v \neq w$ in (1.12), the equations derived above contain the parameter $v - w = u_0$; e.g. the last relation takes the form

$$\begin{aligned} & \check{R}(u, u + u_0) \left(q^{u+u_0} f \otimes k^{\frac{1}{2}} \otimes k^{\frac{1}{2}} + q^{-u} (q^{u_0} k^{-\frac{1}{2}} \otimes f \otimes k^{\frac{1}{2}} + q^{-u_0} k^{-\frac{1}{2}} \otimes k^{-\frac{1}{2}} \otimes f) \right) \\ &= \left(q^{u+u_0} k^{-\frac{1}{2}} \otimes k^{-\frac{1}{2}} \otimes f + q^{-u} (q^{-u_0} k^{-\frac{1}{2}} \otimes f \otimes k^{\frac{1}{2}} + q^{u_0} f \otimes k^{\frac{1}{2}} \otimes k^{\frac{1}{2}}) \right) \check{R}(u, u + u_0). \end{aligned} \quad (1.15)$$

The extension of these equations for the matrices $R^{r'_1 \times r''_1 \ r'_2 \times r''_2}$ acting on the space $[V_{r'_1} \otimes V_{r''_1}] \otimes [V_{r'_2} \otimes V_{r''_2}]$ can be found taking $L^{r'_1} \otimes L^{r''_1}$ instead of L^{r_1} in (1.12).

1.2 Projection operators and indecomposable representations.

At general values of q the tensor product $V_{r_1} \otimes V_{r_2}$ admits Clebsh-Gordan decomposition (1.6), and the eigenvalues c_r of the Casimir operator c are different for different r . It means, that any invariant operator a , $[a, g] = 0$, $g \in sl_q(2)$, acts on each of the irreducible spaces as an identity operator, and hence can be represented as a sum over the projection operators P_r on these spaces:

$$a = \sum_r a_r P_r, \quad P_r P_{r'} = P_r \delta_{rr'}. \quad (1.16)$$

Particularly, $c = \sum_{r=|r_1-r_2|}^{r_1+r_2-1} c_r P_r$. This means, that $\check{R}^{r_1 r_2}$ -matrix ($\check{R}^{r_1 r_2} : V_{r_1} \otimes V_{r_2} \Rightarrow V_{r_2} \otimes V_{r_1}$; when $r_1 \neq r_2$, the relation (1.9) implies $\check{R}^{r_1 r_2} \Delta^{r_1 r_2} = \Delta^{r_2 r_1} \check{R}^{r_1 r_2}$) acquires the form $\check{R}^{r_1 r_2}(u) = \sum_{r=|r_1-r_2|}^{r_1+r_2-1} f_r(u) \check{P}_r$ [10, 9, 17]. Here $\check{P}_r \equiv \mathcal{P}^{r_1 r_2} P_r$, with $\mathcal{P}^{r_1 r_2}$ being an identical transformation operator translating the space $V_{r_1} \otimes V_{r_2}$ into the isomorphic space $V_{r_2} \otimes V_{r_1}$, and $\mathcal{P}^{r \ r'} \mathcal{P}^{r' \ r} = \mathbb{I}^{r \ r}$, $\mathcal{P}^{r \ r} = \mathbb{I}$ (\mathbb{I} is the unity operator defined on the space $V_r \otimes V_r$).

When at least one of the representations V_{r_1} and V_{r_2} is not irreducible, then in the decomposition of their tensor product some irreps have the same eigenvalues of the Casimir operator. Suppose, $R^{r \ r'}(u)$ acts on the tensor product $U_r \otimes U_{r'}$, where U_r or/and $U_{r'}$ are reducible, and it takes place the fusion $U_r \otimes U_{r'} = \bigoplus_{\bar{r}} \bigoplus_i^{\epsilon_{\bar{r}}} V_{\bar{r}}^i$. $\epsilon_{\bar{r}}$ is the multiplicity of the irrep $V_{\bar{r}}$, $\sum_{\bar{r}} \epsilon_{\bar{r}} = r r'$. Here an additional index $i \in \{1, \dots, \epsilon_{\bar{r}}\}$ is attached to distinguish isomorphic irreps $V_{\bar{r}}^i$ corresponding to the same eigenvalue $c_{\bar{r}}$. Then among the invariant operators, commuting with the algebra generators, also projectors $P_{\bar{r}}^{ij}$ appear, which map irreps $V_{\bar{r}}^i$ to each other. So, the R -matrix, as any invariant operator, admits a linear representation over the set of the projectors $P_{\bar{r}}^{ij}$ of number $\sum_{\bar{r}} \epsilon_{\bar{r}}^2$, i.e.

$$\check{R}^{r \ r'}(u) = \mathcal{P}^{r \ r'} \sum_{\bar{r}} \sum_{i,j} f_{\bar{r}}^{ij}(u) P_{\bar{r}}^{ij}, \quad P_{\bar{r}}^{ij} P_{\bar{r}'}^{kr} = P_{\bar{r}}^{ir} \delta_{jk} \delta_{\bar{r} \bar{r}'}. \quad (1.17)$$

At the exceptional values of deformation parameter q , as it was stated, among the representations on which the R -matrix acts also indecomposable representations \mathcal{I} can be included along with the ordinary irreducible representations V . In this case the set of the possible projectors includes also the operators $P' : \mathcal{I} \rightarrow \mathcal{I}$, which are acting inside of the spaces of the indecomposable representations not as unity matrices. The symbolic structure of the indecomposable representation can be shown as $\mathcal{I} = \mathcal{U} \cup \mathcal{U}'$, on which the algebra generators $\{g\}$ act in the following way

$$g \cdot \mathcal{U} \Rightarrow \mathcal{U}, \quad g \cdot \mathcal{U}' \Rightarrow \mathcal{I}. \quad (1.18)$$

The vectors belonging to \mathcal{U}' are defined up to the addition of the vectors belonging to an irreducible representation U ($\dim[\mathcal{U}'] = \dim[U]$), which is the proper subspace of \mathcal{U} and have vectors with zero norm [13, 8]. The action of the Casimir operator on these spaces is given by: $c \cdot \mathcal{U} = c_{\mathcal{I}} \mathbb{I} \cdot \mathcal{U}$, where \mathbb{I} is the unit operator, and $c \cdot \mathcal{U}' = c_{\mathcal{I}} \mathbb{I} \cdot \mathcal{U}' + c'_{\mathcal{I}} \mathbb{I} \cdot U$. Similarly, together with the usual P , acting as unity operator on the indecomposable representation, a projection operator P' , $P' \cdot \mathcal{U} = 0$, $P' \cdot \mathcal{U}' = U$, can be introduced. In the case, when decomposition includes $n \geq 2$ isomorphic indecomposable representations $\mathcal{I}^i = \mathcal{U}^i \cup \mathcal{U}^{\prime i}$, one is able to construct $2n^2$ independent projection operators P^{ij} , $P^{\prime ij}$, $i, j = 1, \dots, n$, acting as

$$\begin{aligned} P^{ij} \cdot \mathcal{I}^k &= \delta_{jk} \mathcal{I}^i, \\ P^{\prime ij} \cdot \mathcal{U}^{k'} &= \delta_{jk} \mathcal{U}^i, \quad P^{\prime ij} \cdot \mathcal{U}^k = 0. \end{aligned} \quad (1.19)$$

The projectors have the following obvious properties

$$P^{ij} P^{kp} = P^{ip} \delta_{jk}, \quad P^{\prime ij} P^{\prime kp} = 0, \quad P^{ij} P^{\prime kp} = P^{\prime ij} P^{kp}. \quad (1.20)$$

Note, that the isomorphic representations having the same dimension, structure and eigenvalues of the Casimir operator, can differ by the signs of the eigenvalues of the generator k , conditioned by the algebra automorphism $k \rightarrow -k$, $e \rightarrow \pm e$, $f \rightarrow \mp f$. The projectors P^{ij} and $P^{\prime ij}$ relate to each other only vectors with the same set of the eigenvalues of k , as it is implied by symmetry. And it means, that for the mentioned situation the action of the projectors P^{ij} , $P^{\prime ij}$ must have slight modification in comparison with (1.19). We shall touch all these aspects in details below for the discussed cases.

1.3 Projectors and Casimir operator.

In this subsection we want to present another approach to the problem. Let us be given a set of the algebra representations $\mathcal{S} = \{V, \mathcal{I}\}$ and let us consider on this set a general matrix, which is commutative with the algebra. The number of degrees of freedom of this matrix is given by the number of the mutually linear independent matrices (basis matrices) which are invariant with respect to the symmetry algebra. We can choose as the basis matrices the projection operators described above, i.e. the operators which act non-trivially (are not zero) only on one non-reducible space, mapping the latter either to itself or to another non-reducible space. Note, that each invariant operator on \mathcal{S} , including the identity and Casimir operators, can be represented as a linear superposition of these operators. Now we discuss the inverse problem: how the projection operators can be built by means of the Casimir and unity operators.

The case (1.16) discussed in the beginning of the previous section corresponds to $\mathcal{S} = V_{r_1} \otimes V_{r_2}$ (1.6), and the projectors P_r , as it is well known, are given by polynomials of degree $r_1 + r_2 - 1$ in terms of the Casimir operator c , as the eigenvalues c_r at general q do not coincide one with other:

$$P_r = \prod_{p \neq r} \frac{c - c_p \mathbb{I}}{c_r - c_p}. \quad (1.21)$$

Let us now consider some particular cases, when \mathcal{S} contains indecomposable representations. If it consists of a single indecomposable representation $\mathcal{S} = \mathcal{I}$, then

$$c = c_{\mathcal{I}} P_{\mathcal{I}} + c'_{\mathcal{I}} P'_{\mathcal{I}}, \quad P_{\mathcal{I}} = \mathbb{I}, \quad P'_{\mathcal{I}} = \frac{c - c_{\mathcal{I}} \mathbb{I}}{c'_{\mathcal{I}}}. \quad (1.22)$$

When $\mathcal{S} = \mathcal{I} \oplus V_r$, one has

$$\begin{aligned} c &= c_{\mathcal{I}} P_{\mathcal{I}} + c'_{\mathcal{I}} P'_{\mathcal{I}} + c_r P_r, & \mathbb{I} &= P_{\mathcal{I}} + P_r, \\ P'_{\mathcal{I}} &= \left(\frac{c - c_{\mathcal{I}} \mathbb{I}}{c'_{\mathcal{I}}} \right) \left(\frac{c - c_r \mathbb{I}}{c_{\mathcal{I}} - c_r} \right), \\ P_{\mathcal{I}} &= \left(\frac{c - (2c_{\mathcal{I}} - c_r) \mathbb{I}}{c_r - c_{\mathcal{I}}} \right) \left(\frac{c - c_r \mathbb{I}}{c_{\mathcal{I}} - c_r} \right), & P_r &= \left(\frac{c - c_{\mathcal{I}} \mathbb{I}}{c_r - c_{\mathcal{I}}} \right)^2. \end{aligned} \quad (1.23)$$

The next simple case is $\mathcal{S} = \mathcal{I}_1 \oplus \mathcal{I}_2$, $c_{\mathcal{I}_1} \neq c_{\mathcal{I}_2}$. Here the following formulas take place:

$$\begin{aligned} c &= c_{\mathcal{I}_1} P_{\mathcal{I}_1} + c'_{\mathcal{I}_1} P'_{\mathcal{I}_1} + c_{\mathcal{I}_2} P_{\mathcal{I}_2} + c'_{\mathcal{I}_2} P'_{\mathcal{I}_2}, \\ P'_{\mathcal{I}_i} &= \left(\frac{c - c_{\mathcal{I}_i} \mathbb{I}}{c'_{\mathcal{I}_i}} \right) \left(\frac{c - c_{\mathcal{I}_j} \mathbb{I}}{c_{\mathcal{I}_i} - c_{\mathcal{I}_j}} \right)^2, \quad i = 1, 2, \quad j \neq i, \\ P_{\mathcal{I}_i} &= \left(\frac{2c - (3c_{\mathcal{I}_i} - c_{\mathcal{I}_j}) \mathbb{I}}{c_{\mathcal{I}_j} - c_{\mathcal{I}_i}} \right) \left(\frac{c - c_{\mathcal{I}_j} \mathbb{I}}{c_{\mathcal{I}_i} - c_{\mathcal{I}_j}} \right)^2, \quad i = 1, 2, \quad j \neq i. \end{aligned} \quad (1.24)$$

Above formulas have obvious generalizations for the set $\mathcal{S} = V_{r_1} \oplus \cdots \oplus V_{r_n} \oplus \mathcal{I}_1 \oplus \cdots \oplus \mathcal{I}_p$, where all the representations have different eigenvalues of c :

$$\begin{aligned}
c &= \sum_{i=1}^n c_{r_i} P_{r_i} + \sum_{j=1}^p (c'_{\mathcal{I}_j} P'_{\mathcal{I}_j} + c_{\mathcal{I}_j} P_{\mathcal{I}_j}) \\
P_{r_k} &= \prod_{i \neq k}^n \left(\frac{c - c_{r_i} \mathbb{I}}{c_{r_k} - c_{r_i}} \right) \prod_j^p \left(\frac{c - c_{\mathcal{I}_j} \mathbb{I}}{c_{r_k} - c_{\mathcal{I}_j}} \right)^2, \\
P'_{\mathcal{I}_k} &= \frac{c - c_{\mathcal{I}_k} \mathbb{I}}{c'_{\mathcal{I}_k}} \prod_i^n \left(\frac{c - c_{r_i} \mathbb{I}}{c_{\mathcal{I}_k} - c_{r_i}} \right) \prod_{j \neq k}^p \left(\frac{c - c_{\mathcal{I}_j} \mathbb{I}}{c_{r_k} - c_{\mathcal{I}_j}} \right)^2, \\
P_{\mathcal{I}_k} &= (c_{V\mathcal{I}} c - \bar{c}_{V\mathcal{I}} \mathbb{I}) \prod_i^n \left(\frac{c - c_{r_i} \mathbb{I}}{c_{\mathcal{I}_k} - c_{r_i}} \right) \prod_{j \neq k}^p \left(\frac{c - c_{\mathcal{I}_j} \mathbb{I}}{c_{r_k} - c_{\mathcal{I}_j}} \right)^2, \\
c_{V\mathcal{I}} &= \sum_i^n \frac{1}{c_{r_i} - c_{\mathcal{I}_k}} + \sum_{j \neq k}^p \frac{2}{c_{\mathcal{I}_j} - c_{\mathcal{I}_k}}, \quad \bar{c}_{V\mathcal{I}} = c_{V\mathcal{I}} c_{\mathcal{I}_k} - 1.
\end{aligned} \tag{1.25}$$

How should be generalized the above formulas in case of degeneracy of the Casimir operator? The answer seems to be simple: when the eigenvalues spectrum of c has degeneracy of degree n then one should consider an operator $c^{\frac{1}{n}}$ instead of c ($(c^{\frac{1}{n}})^n = c$), eigenvalues' spectrum of which is not degenerated and one can use the formula (1.26), replacing c with $c^{\frac{1}{n}}$ and with its eigenvalues. A detailed consideration is placed in the Appendix.

2 Solutions to the YBE

The solutions $\check{R}^{r_1 r_2}$ to the YBE, when V_{r_1} and V_{r_2} are irreps, for the quantum super-algebra $osp_q(1|2)$ at general q are considered in [17]. As there is a full one-to-one correspondence between the representations of two quantum algebras at general q [14, 15, 8], we can take the solutions given there and verify, that after the appropriate change of the quantum deformation parameter, and after removing the signs connected with the gradings, we shall arrive at the solutions to the YBE for $sl_q(2)$.

Let us briefly represent all the solutions to the YBE at general q for inhomogeneous spectral parameter dependent $\check{R}^{r_1 r_2}(u)$ -matrix. From Jimbo's relations (1.8) one finds (below $r_1 = 2j_1 + 1$, $r_2 = 2j_2 + 1$)

$$\check{R}^{(r_1 r_2)}(u) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \mathbf{r}_j(u) \check{P}_{2j+1}, \tag{2.1}$$

$$\mathbf{r}_{j'}(u) = \prod_{j=j'}^{j_1+j_2-1} \left[\Upsilon_{j_1 j_2}^j \frac{q^u - q^{-u} q^{2(j'+1)}}{q^{-u} - q^u q^{2(j'+1)}} \right] \mathbf{r}_{j_1+j_2}(u), \tag{2.2}$$

$$\Upsilon_{j_1 j_2}^j = q^{i_2 - i_1} \frac{\alpha_{j_2}^{j-i_1}}{\alpha_{j_1}^{j-i_2}} \frac{C\left(\begin{smallmatrix} j_1 & j_2 & j \\ i_1 & j-i_1 & j \end{smallmatrix}\right) C\left(\begin{smallmatrix} j_2 & j_1 & j+1 \\ i_2 & j+1-i_2 & j+1 \end{smallmatrix}\right)}{C\left(\begin{smallmatrix} j_1 & j_2 & j+1 \\ i_1 & j+1-i_1 & j+1 \end{smallmatrix}\right) C\left(\begin{smallmatrix} j_2 & j_1 & j \\ i_2 & j-i_2 & j \end{smallmatrix}\right)}. \tag{2.3}$$

where the projector operators \check{P}_r , $\check{P}_r \cdot V_g = \delta_{rg} V_g$, are acting as map $V_{2j_1+1} \otimes V_{2j_2+1} \rightarrow V_{2j_2+1} \otimes V_{2j_1+1}$. When $r_1 = r_2$, then $\check{P}_r = P_r$ and $\Upsilon_{j_1 j_2}^j = 1$ [10, 9, 13]. By the notations $C \begin{pmatrix} j_1 & j_2 & j \\ i_1 & i_1 & i \end{pmatrix}$ we have denoted the Clebsh-Gordan coefficients and the parameters α_j^i are the matrix elements of the algebra generator e on the vector space $V_{2j+1} = \{[v_i]_j, i = -j, -j+1, \dots, j\}$: $e \cdot [v_i]_j = \alpha_j^i [v_{i+1}]_j$, $k \cdot [v_i]_j = q^{2i} [v_i]_j$. The expression (2.3) is the same for all permissible values of i_1 and i_2 from the range $-j_1 \leq i_1 \leq j_1, -j_2 \leq i_2 \leq j_2$ (see [8, 17]).

By means of Jimbo's ordinary relations (1.8) or the relations for composite matrices (1.13, 1.14) we can find solutions to the YBE with $\check{R}^{r_1 r'_2 \times r''_2} (\check{R}^{r'_1 \times r''_1 r'_2 \times r''_2})$. These relations are inherited from the Lax representations of the YBE (1.10, 1.12) and their solutions can be obtained by the descendant procedure from the fundamental solution $R^{2,2}(u)$ [9]. By this reason, as we shall see, at roots of unity solving all Jimbo's relations leads to the solutions being the limit cases of those existing at general q (like the fundamental solution). So, at roots of unity for obtaining essentially new solutions to the YBE one must consider directly the YBE. Note, although, that (as we shall see later on, in Section 2.2) using only one pair of Jimbo's composite relations (namely, (1.14), and its analogue for the generator e) will bring at roots of unity to some definite generalizations of the solutions existing at general q .

At general q also there are solutions to the YBE which do not admit Lax representation (i.e. do not obey the relations (1.10)). When $r_1 = r_2 = 3$ besides of the solution $\check{R}_1^{3,3}(u)$, which can be obtained from the general solution (2.2), there is a separate solution $\check{R}_2^{3,3}(u)$, which does not admit descendant solutions $R^{3r_i}, R^{r_j r_i}$ for higher r_i (see [12], [17]). Below there is done a multiplicative transformation of the spectral parameter of $\check{R}_1^{3,3}(u)$ in comparison with (2.2), $u \rightarrow -u/2$:

$$\check{R}_1^{3,3}(u) = P_5 + \frac{q^{4+u}-1}{q^4-q^u} P_3 + \frac{(q^{2+u}-1)(q^{4+u}-1)}{(q^2-q^u)(q^4-q^u)} P_1, \quad \check{R}_2^{3,3}(u) = P_5 + \frac{q^4 q^u - 1}{q^4 - q^u} P_3 + \frac{q^6 q^u + 1}{q^6 + q^u} P_1. \quad (2.4)$$

Also there is another solution, which does not obey (1.8), and which does not distinguish the projectors P_5 and P_3 , namely

$$\check{R}_{\pm}^{3,3}(u) = P_5 + P_3 + \frac{a_{\pm} + q^u}{1 + a_{\pm} q^u} P_1, \quad (2.5)$$

$$a_{\pm} = \frac{-1}{2q^4} \left(1 + 2q^2 + q^4 + 2q^6 + q^8 \pm (1 + q^2 + q^4) \sqrt{1 + 2q^2 - q^4 + 2q^6 + q^8} \right).$$

Note, that $a_+ a_- = 1$ and hence $\check{R}_+^{3,3}(u) = \check{R}_-^{3,3}(-u)$. This solution belongs to the series of the R^{rr}

solutions which admit "baxterized" [21] form $R = q^u R^+ + q^{-u} R^-$,

$$\check{R}^{r_1 r_2}(u) = \mathbb{I} + \left(\frac{a + q^u}{1 + a q^u} - 1 \right) P_1, \quad a = \frac{i + \sqrt{-1 + 4/[r]_q^2}}{-i + \sqrt{-1 + 4/[r]_q^2}}.$$

Here \mathbb{I} is the $r^2 \times r^2$ unity matrix defined on the space $V^r \times V^r$. There is no generalization $\check{R}^{r_1 r_2}(u)$ for such matrices in the case of $r_1 \neq r_2$. At $r = 2$ (2.5) coincides with the fundamental solution in (2.1).

2.1 YBE solutions \check{R}_{VV} : $\check{R}^{33}(u)$ and some notes and statements.

Solutions at $q^3 = \pm 1$. As an illustrative example we consider here the case $\mathcal{N} = 3$, which will provide us with the characteristic properties of the solutions \check{R}_{VV} at roots of unity.

At $q^3 = \pm 1$ the existing non-reducible representations of the algebra $sl_q(2)$ are the irreps V_2, V_3 (for the super-algebra $osp_q(1|2)$ the fundamental representation is the V_3) and the indecomposable representations $\mathcal{I}_{\{4,2\}}^{(6)}$ and $\mathcal{I}_{\{5,1\}}^{(6)}$. Particularly, the tensor products at general q , $V_3 \otimes V_2 = V_4 \oplus V_2$ and $V_3 \otimes V_3 = V_5 \oplus V_3 \oplus V_1$, degenerate and turn correspondingly into $\mathcal{I}_{\{4,2\}}^{(6)}$ and $\mathcal{I}_{\{5,1\}}^{(6)} \oplus V_3$ at $q^3 = \pm 1$.

The simplest cases for which we can try to find the solutions correspond to the matrices $\check{R}^{33}(u)$ and $\check{R}^{32}(u)/\check{R}^{23}(u)$. The spectral parameter dependent solution $\check{R}^{23}(u)$ to the YBE ($\check{R}^{22}\check{R}^{23}\check{R}^{23} = \check{R}^{23}\check{R}^{23}\check{R}^{22}$) at general q is unique (2.1), which is fixed by the fundamental matrix $\check{R}^{22}(u)$. If to take as $\check{R}^{22}(u)$ the unity matrix or any other $sl_q(2)$ -symmetric 4×4 matrix, then the solution $\check{R}^{23}(u)$ is constant. The same is valid at $q^3 = \pm 1$ as well, when the decomposition $\check{R}^{23}(u) = \check{P}_4 + f(u)\check{P}_2$ smoothly transforms into $\check{P}_{\mathcal{I}_{\{4,2\}}^{(6)}} + \bar{f}(u)\check{P}'_{\mathcal{I}_{\{4,2\}}^{(6)}}$ (see the analysis in the previous section). Here $\check{P}_{\mathcal{I}_{\{4,2\}}^{(6)}} = \mathbb{I}$, $\check{P}'_{\mathcal{I}_{\{4,2\}}^{(6)}} = \lim_{q \rightarrow e^{i\frac{r\pi}{3}}} (c_4 - c_2)\check{P}_2$ and $\bar{f}(u) = \lim_{q \rightarrow e^{i\frac{r\pi}{3}}} (f(u) - 1)/(c_4 - c_2)$, $r = 1, 2, 4, 5$.

Similarly we must take $\check{R}^{33}(u)$ at $q^3 = \pm 1$ in the form of $\check{R}^{33}(u) = P_{\mathcal{I}_{\{5,1\}}^{(6)}} + f(u)P'_{\mathcal{I}_{\{5,1\}}^{(6)}} + g(u)P_3$. The Casimir operator on the space of the tensor product $V_3 \otimes V_3$ can be expressed as $c^{33} = \frac{-1}{3}P_{\mathcal{I}^{(6)}} + P'_{\mathcal{I}^{(6)}} + \frac{2}{3}P_3$, and $P_{\mathcal{I}^{(6)}} + P_3 = \mathbb{I}$. The projectors P_5 and P_1 have poles at $q^3 = \pm 1$, but the solutions (2.4, 2.5) are well defined and are transformed into the following expressions (we have fixed below $q = (-1)^{1/3} = e^{i\pi/3}$)

$$\check{R}_1^{33}(u) = P_{\mathcal{I}_{\{5,1\}}^{(6)}} + \frac{i\sqrt{3}(q^{2u}-1)}{1+q^u+q^{2u}}P'_{\mathcal{I}_{\{5,1\}}^{(6)}} + \frac{q^{u+1}+1}{q+q^u}P_3, \quad \check{R}_2^{33}(u) = P_{\mathcal{I}_{\{5,1\}}^{(6)}} + \frac{i\sqrt{3}(q^u-1)}{1+q^u}P'_{\mathcal{I}_{\{5,1\}}^{(6)}} + \frac{q^{u+1}+1}{q+q^u}P_3, \quad (2.6)$$

$\check{R}_{\pm}^{33} = \mathbb{I} \pm \frac{i(q^u - 1)}{1 + q^u} P'_{\mathcal{I}_{\{5,1\}}^{(6)}}$. There are not new constant or spectral parameter dependent solutions at roots of unity also for the YBE with $\check{R}^{32}(u)$ matrix ($\check{R}^{33} \check{R}^{32} \check{R}^{32} = \check{R}^{32} \check{R}^{32} \check{R}^{33}$). The only spectral parameter solutions are the limit cases of the corresponding solutions (2.1). If to take in the YBE as $\check{R}^{33}(u)$ any other $sl_q(2)$ -invariant 9×9 matrix, the $\check{R}^{32}(u)$ -matrix becomes constant (equivalent to the constant solution $\check{R}^{23}(u)$).

The solution at $q^6 = -1$. Note, that all of the spectral parameter dependent solutions discussed up to now are supplemented by the normalization condition $\check{R}(0) = \mathbb{I}$. We would like to mention a peculiarity which is met at $q^6 = -1$ ($t^3 = 1$ for $osp_t(1|2)$ [17]). Here there is no degeneration in the fusion for the tensor product $V_3 \otimes V_3$, but the following solution to the YBE [17]

$$q^6 = -1, \quad \check{R}_o^{33}(u) = P_5 + \frac{q^4 q^u - 1}{q^4 - q^u} P_3 - P_1. \quad (2.7)$$

has the property $\check{R}_o^{33}(0) = P_5 + P_3 - P_1$. At first sight this solution coincides with the solution $\check{R}_2^{33}(u)$ in (2.4), if to take the limit $q \rightarrow (-1)^{r/6}$, $r = 1, 3, 5, 7, 9, 11$. But there is a notable difference at the point $u = 0$, where both of $\check{R}_{1,2}^{33}(0)$ (2.4) become unity matrices, which is important. It means, that $\lim_{q \rightarrow (-1)^{r/6}} \lim_{u \rightarrow 0} \check{R}_2^{33}(u) \neq \lim_{u \rightarrow 0} \lim_{q \rightarrow (-1)^{r/6}} \check{R}_2^{33}(u)$. Note, that for $q^4 = 1$ the matrix \check{R}_o is a solution too (and the peculiarities noted above about the not-coinciding limits are right also here), but as we know for this case V_3 is not an irrep. We can denote it as a $\bar{V}_3 \supset V_1$ (as in [8]) and write the proper fusion $\bar{V}_3 \otimes \bar{V}_3 = \mathcal{I}_{\{5,3\}}^{(8)} \oplus V_1$, where $\mathcal{I}_{\{5,3\}}^{(8)}$ is equivalent to the direct sum of two $\mathcal{I}_{\{3,1\}}^{(4)}$. We shall not analyze this case, as it is included in a non-direct way in consideration of $\bigotimes^4 V_2 = \mathcal{I}_{\{3,1\}}^{(4)} \otimes \mathcal{I}_{\{3,1\}}^{(4)}$ (as $\mathcal{I}_{\{3,1\}}^{(4)} \supset \bar{V}_3$ ([8])) done further in this section.

Some notes and statements. The expressions above (2.6) can be obtained either by direct solving of the YBE at roots of unity or by taking the corresponding limits of the solutions existing at general q , using appropriate modifications of the expressions. When at $q^n = 1$ in the fusion of two irreps indecomposable representation $\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}$ arises from the merging of the representations V_r and $V_{\mathcal{R}-r}$, and the projectors $P_{\mathcal{R}-r}$ and P_r acquire singularities [8], the Casimir operator remains well defined and can be rewritten in terms of the projectors $P_{\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}}$ and $P'_{\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}}$. As at general q the projectors $P_{\mathcal{R}-r}$ and P_r are included in c as the sum $c_{\mathcal{R}-r} P_{\mathcal{R}-r} + c_r P_r$, we can rewrite it as $c_r (P_r + P_{\mathcal{R}-r}) + (c_{\mathcal{R}-r} - c_r) P_{\mathcal{R}-r}$, where the first summand $P_r + P_{\mathcal{R}-r}$ transforms at roots of unity to the projector $P_{\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}}$ and the second one to the projector $(c_{\mathcal{R}-r} - c_r) / c_r P_{\mathcal{R}-r} \Rightarrow P'_{\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}}$. At the

given roots of unity the Casimir operator becomes degenerate, $c_{\mathcal{R}-r} = c_r$, and here the singularity in the projector $P_{\mathcal{R}-r}$ has been canceled by the zero in the nominator. Putting in the expression of the matrix $\check{R}_{VV}(u)$ the projectors $P_{\mathcal{R}-r}$ and P_r written in terms of $P_{\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}}$ and $P'_{\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})}}$, and then taking the corresponding values of q we shall obtain the exact well-defined expression. This is conditioned by the fact, that the coefficients of the projectors $P_{\mathcal{R}-r}$ and P_r in the expansion of $\check{R}_{VV}(u)$ (2.2) coincide at the corresponding roots of unity, as it was for the case of the Casimir operator.

Essentially new solutions to the YBE can be obtained in the cases, when the number of the projectors at roots of unity increases comparing with the case of general q . It happens when we consider matrices $\check{R}_{V\mathcal{I}}$ and $\check{R}_{\mathcal{I}\mathcal{I}}$ acting on the tensor products $V_r \otimes \mathcal{I}_{\{r', \mathcal{R}-r'\}}^{(\mathcal{R})}$ and $\mathcal{I}_{\{r, \mathcal{R}-r\}}^{(\mathcal{R})} \otimes \mathcal{I}_{\{r', \mathcal{R}-r'\}}^{(\mathcal{R})}$, which stand instead of $V_r \otimes (V_{r'} \oplus V_{\mathcal{R}-r'})$ and $(V_r \oplus V_{\mathcal{R}-r}) \otimes (V_{r'} \oplus V_{\mathcal{R}-r'})$ at general q . We shall analyse the simplest such case below, when $q = i$. We can calculate that the number of the linear independent $r\mathcal{R} \times r\mathcal{R}$ - and $\mathcal{R}^2 \times \mathcal{R}^2$ -matrices (hence, the number of the independent projectors also) acting on the $r\mathcal{R}$ and \mathcal{R}^2 -dimensional representation spaces of the mentioned tensor products at general q and at roots of unity ($q^{\mathcal{R}}=1$) are different. Hereafter we shall refer as new solutions (providing q is a root of unity) to those, which are not obtained at roots of unity from the solutions existing at general q .

2.2 YBE solutions at $q = i$.

At $q^4 = 1$ (we fix $q = i$, the case of $q = -i$ is completely equivalent to this case) only two non-reducible highest weight representations exist in the fusions of the fundamental two-dimensional spin-1/2 representations. They are two-dimensional spin-1/2 irrep V_2 and four-dimensional indecomposable representation $\mathcal{I}_{\{3,1\}}^{(4)} = V_2 \otimes V_2$. The tensor product decomposition rules for them have the following form

$$\otimes^2 V_2 = \mathcal{I}_{\{3,1\}}^{(4)}, \quad V_2 \otimes \mathcal{I}_{\{3,1\}}^{(4)} = \oplus^4 V_2, \quad \otimes^2 \mathcal{I}_{\{3,1\}}^{(4)} = \oplus^4 \mathcal{I}_{\{3,1\}}^{(4)}. \quad (2.8)$$

The corresponding YBE for the matrices $R^{2\ 2}$, $\check{R}^{2\ 4}$ and $\check{R}^{4\ 4}$ are

$$\left(\check{R}^{2\ 2}(u) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \check{R}^{2\ 2}(u+v)\right) \left(\check{R}^{2\ 2}(v) \otimes \mathbb{I}\right) = \left(\mathbb{I} \otimes \check{R}^{2\ 2}(v)\right) \left(\check{R}^{2\ 2}(u+v) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \check{R}^{2\ 2}(v)\right), \quad (2.9)$$

$$\left(\check{R}^{2\ 2}(u) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \check{R}^{2\ 4}(u+v)\right) \left(\check{R}^{2\ 4}(v) \otimes \mathbb{I}\right) = \left(\mathbb{I} \otimes \check{R}^{2\ 4}(v)\right) \left(\check{R}^{2\ 4}(u+v) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \check{R}^{2\ 2}(v)\right), \quad (2.10)$$

$$\left(\check{R}^{4\ 4}(u) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \check{R}^{4\ 4}(u+v)\right) \left(\check{R}^{4\ 4}(v) \otimes \mathbb{I}\right) = \left(\mathbb{I} \otimes \check{R}^{4\ 4}(v)\right) \left(\check{R}^{4\ 4}(u+v) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \check{R}^{4\ 4}(u)\right), \quad (2.11)$$

acting accordingly on the vector spaces $V_2 \otimes V_2 \otimes V_2$, $V_2 \otimes V_2 \otimes \mathcal{I}_{\{3,1\}}^{(4)}$ and $\mathcal{I}_{\{3,1\}}^{(4)} \otimes \mathcal{I}_{\{3,1\}}^{(4)} \otimes \mathcal{I}_{\{3,1\}}^{(4)}$. Here we have preferred to write the action of the operators in the tensor product form to avoid the usual lower indexes (see e.g. Eq. (1.7)), which distinguish different spaces, meanwhile the indexes used here denote the dimensions of the representation spaces.

Note, that also the YBE defined on the space $\mathcal{I}_{\{3,1\}}^{(4)} \otimes \mathcal{I}_{\{3,1\}}^{(4)} \otimes V_2$ could be considered,

$$\left(\check{R}^{4,4}(u) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \check{R}^{4,2}(u+v)\right) \left(\check{R}^{4,2}(v) \otimes \mathbb{I}\right) = \left(\mathbb{I} \otimes \check{R}^{4,2}(v)\right) \left(\check{R}^{4,2}(u+v) \otimes \mathbb{I}\right) \left(\mathbb{I} \otimes \check{R}^{4,4}(v)\right), \quad (2.12)$$

the solutions of which are not necessarily the solutions to the equations (2.10) and (2.11). Here we shall concentrate on the YBE (2.10) and (2.11).

There is a unique non-trivial solution $R^{2,2}(u)$ to (2.9), which is just the limit $q \rightarrow i$ of the solution (2.1), $\check{R}^{2,2}(u) = \mathbb{I} + \frac{i(1-e^u)}{1+e^u} c^{2,2}$ (we have chosen the parametrization taking into account the freedom of the normalization of the spectral parameter, to replace q^u with $\exp(u)$, which is a convenient expression for the fixed values of q). $R^{2,2}(u)$ can be expressed also by means of two projection operators, $P_{\mathcal{I}_{\{3,1\}}^{(4)}} (= \mathbf{I})$ and $P'_{\mathcal{I}_{\{3,1\}}^{(4)}} (\approx \lim_{q \rightarrow i} (c_3 - c_1) P_1)$.

2.2.1 The solutions $\check{R}^{2,4}(u)$.

The two-dimensional spaces in the decomposition of $V_2 \otimes \mathcal{I}_{\{3,1\}}^{(4)}$ (2.8) must be considered pairwise, \tilde{V}_2^i , $i = 1, 2$ (two representations, emerging from the splitting of the representation V_4 in $\bigotimes^3 V_2$ at $q = i$) and the remaining two V_2^i , $i = 1, 2$: $V_2 \otimes V_2 \otimes V_2 = V_4 \oplus V_2 \oplus V_2 \Rightarrow_{q \rightarrow i} \tilde{V}_2 \oplus \tilde{V}_2 \oplus V_2 \oplus V_2$, as they have Casimir eigenvalues c_4 , c_2 differing by sign at $q = i$. Thus the projection operators now are eight, \tilde{P}_2^{ij} and P_2^{ij} , $i, j = 1, 2$ (at general q they are five, P_4 and P_2^{ij} , $i, j = 1, 2$). As here we have larger space of the projectors than for the case of general q , we can look for new solutions in the form

$$R^{2,4}(u) = \sum_{i,j=1,2} \left(\tilde{f}_{ij}(u) \tilde{P}_2^{ij} + f_{ij}(u) P_2^{ij} \right). \quad (2.13)$$

Taking in the YBE (2.10) the intertwiner $\check{R}^{2,2}(u) = \mathbb{I} + \frac{i(1-e^u)}{1+e^u} c^{2,2}$, we find that the only spectral parameter dependent solution of $\check{R}^{2,4}(u)$ with the normalization property $\check{R}^{2,4}(0) = \mathbb{I}$, is given as follows

$$\check{R}^{2,4}(u) = \left[\tilde{P}_2^{11} + \tilde{P}_2^{22} \right] + \frac{1 + 6e^u + e^{2u}}{2(1 + e^u)^2} \left[P_2^{11} + P_2^{22} \right] + \frac{i(e^u - 1)}{2(1 + e^u)^2} \left[P_2^{12}(1 + 3e^u) + P_2^{21}(3 + e^u) \right]. \quad (2.14)$$

This matrix corresponds to the ordinary XX model. It is just the composite solution $\check{R}^{24}(u) = (R^{22}(u) \otimes \mathbf{I})(\mathbf{I} \otimes R^{22}(u))$ at $q = i$. Such solution could be obtained also from Jimbo's composite relations (1.13, 1.14). The relation (1.15) provides with the solution $(R^{22}(u) \otimes \mathbf{I})(\mathbf{I} \otimes R^{22}(u + u_0))$ at general q and in the limit $q \rightarrow i$, too. At $q = i$ there is also another generalization of the matrix (2.14), for which $R^{24}(0) \neq \mathbb{I}$, and where the projectors \tilde{P}_2^{11} and \tilde{P}_2^{22} have different coefficient functions containing an arbitrary parameter f_0 . This means that such solution could not exist at general q , as in the limit $q \rightarrow i$ the projectors \tilde{P}_2^{ij} appear only in the following sum, $P_4 \Rightarrow \tilde{P}_2^{11} + \tilde{P}_2^{22}$. The general expression of that solution is the following

$$\begin{aligned} \check{R}^{24}(u; u_0, f_0) = & 2 \left((1 + f_0)(1 + \cosh[u_0]) + \cosh[u] + \cosh[u + u_0] + (1 - f_0) \sinh[u_0] \right) \tilde{P}_2^{11} + \\ & 2 \left((1 - f_0)(1 + \cosh[u_0]) + \cosh[u] + \cosh[u + u_0] + (1 + f_0) \sinh[u_0] \right) \tilde{P}_2^{22} + \\ & \left(4 - f_0 + \cosh[u] + (2 - 3f_0) \cosh[u_0] + \cosh[u + u_0] + 3f_0 \sinh[u_0] \right) P_2^{11} + \\ & \left(4 + 3f_0 + \cosh[u] + (2 + f_0) \cosh[u_0] + \cosh[u + u_0] - f_0 \sinh[u_0] \right) P_2^{22} + \\ & i \left(f_0 + \cosh[u] - (2 + f_0) \cosh[u_0] + \cosh[u + u_0] + f_0 \sinh[u_0] + 2 \sinh[u + u_0] + 2 \sinh[u] \right) P_2^{12} - \\ & i \left(f_0 + \cosh[u] - (2 + f_0) \cosh[u_0] + \cosh[u + u_0] + f_0 \sinh[u_0] - 2 \sinh[u + u_0] - 2 \sinh[u] \right) P_2^{21}. \end{aligned} \quad (2.15)$$

When $f_0 = 0$ and $u_0 = 0$ this expression coincides with the solution (2.14), after multiplying by an overall function. This expression is a solution to the YBE, and also obeys to (1.15), but the generalization for $w = u + u_0$ of the next composite relation (1.13) fixes $f_0 = 0$.

The other spectral parameter dependent solution, which exists at general q is the representation of the matrix $\check{R}^{23}(u)$ in the space $V_2 \otimes V_2 \otimes V_2$, which we shall denote as $\check{R}^{24(3)}(u)$. This is the solution of Jimbo's ordinary relation (1.8). This solution also contains an arbitrariness coming from the combination of the projectors $\sum' P_2^{ij} (= q/\sqrt{1+q^2+q^4} P_2^{11} - \sqrt{1+q^2+q^4}/q P_2^{22} + q^2 P_2^{12} - q^{-2} P_2^{21})$ in the basis fixed by us) which vanishes after multiplication by the operators containing P_3 , $[P_3 \otimes \mathbb{I}] \sum' P_2^{ij} [\mathbb{I} \otimes P_3] = 0$. Actually it is proportional to the matrix $[P_1 \otimes \mathbb{I}][\mathbb{I} \otimes P_1]$. Here P_1 and P_3 are the 4×4 projector operators into the one- and three- dimensional spaces in the fusion at general q ($V_2 \otimes V_2 = V_1 \oplus V_3$), \mathbb{I} is the 2×2 unity matrix. A term $f(u) \sum' P_2^{ij}$ with arbitrary coefficient function $f(u)$ can be added to $\check{R}^{24(3)}(u)$, and it will remain as a solution to the YBE or Jimbo's ordinary relation (at any values of q). We learn also from these examples, that the existence of the arbitrary functions in the solutions can speak about the possibility to reduce the YBE on the subspaces of the representations (for the given example two separate parts of the

matrix $R^{24}(u)$ are acting separately on the subspace $V_2 \otimes V_3$ and the subspace $V_2 \otimes V_1$ of the entire space $V_2 \otimes V_2 \otimes V_2 = V_2 \otimes (V_3 \oplus V_1)$.

At $q = i$ this solution contains a singularity, and if to take the limit $q \rightarrow i$ after multiplying by $(1 + q^2)$, the solution becomes constant one. One can note that the singular term is proportional to the matrix $\sum' P_2^{ij}$, so by adding to this solution a matrix $\sum' P_2^{ij}$ with appropriate defined coefficient function, we can remove the singularity and have a good defined limit $q \rightarrow i$ (below $f(u)$ is an arbitrary function and $P_2^{11} + P_2^{22} + iP_2^{12} - iP_2^{21} = \sum' P_2^{ij}$)

$$\check{R}^{24(3)}(u) = (\tilde{P}_2^{11} + \tilde{P}_2^{22}) + i \frac{1 + e^{u+u_0}}{e^{u+u_0} - 1} (P_2^{12} + P_2^{21}) + f(u)(P_2^{11} + P_2^{22} + iP_2^{12} - iP_2^{21}). \quad (2.16)$$

Here u_0 is an arbitrary number: the shifting of the spectral parameter is a permissible transformation of the solutions.

We see, as it was expected, that the consideration of Jimbo's relations gives only particular solutions, so in the following we shall deal straightforwardly with the YBE (2.10) and (2.11).

There are numerous constant solutions to (2.10) at $q = i$. Some of them are the limit cases of the spectral parameter dependent solutions taken at $u \rightarrow 0, \pm\infty$. We would like to present below only such solutions, which could be considered as new ones (with existence of \tilde{P}_2^{ij} projectors with different coefficients). Such constant solutions $\check{R}_c^{24}(u)$ are

$$\begin{aligned} \check{R}_c^{24}(u) &= \tilde{P}_2^{22} + g_0 P_2^{11} + \frac{g_0 - 2}{2g_0 - 1} \left(g_0 \tilde{P}_2^{11} + P_2^{22} \right), \\ \check{R}_c^{24}(u) &= \tilde{P}_2^{22} + \frac{g_0^2 - 2(f_0 + g_0)}{2(f_0 + g_0) - 1} \tilde{P}_2^{11} + g_0 P_2^{11} + \frac{g_0 - 2(f_0 g_0 + 2)}{2(f_0 + g_0) - 1} P_2^{22} + f_0 (P_2^{11} + P_2^{22} + iP_2^{12} - iP_2^{21}), \\ \check{R}_c^{24}(u) &= g_0 \left(\tilde{P}_2^{11} - \tilde{P}_2^{22} + \tilde{P}_2^{22} - P_2^{11} \right) + f_0 (P_2^{11} + P_2^{22} + iP_2^{12} - iP_2^{21}). \end{aligned} \quad (2.17)$$

Here g_0 and f_0 are arbitrary constants. And, moreover, all these matrices satisfy to the YBE (2.10) with arbitrary $sl_i(2)$ invariant $\check{R}^{22}(u)$, i.e. $\check{R}^{22}(u) = \mathbb{I} + f(u)c^{22}$, where $f(u)$ can be any function. Spectral parameter dependent solutions with the arbitrary $R^{22}(u)$ -matrix also exist (with \tilde{P}_2^{12} or \tilde{P}_2^{21}),

$$R^{24}(u) = \tilde{P}_2^{12/21} + g(u)(P_2^{11} + P_2^{22} + iP_2^{12} - iP_2^{21}). \quad (2.18)$$

The second part of this solution with arbitrary function $g(u)$ is a constant solution also at general q (i.e. the matrix $\sum' P_2^{ij}$).

Also we would like to mention the following two solutions,

$$\check{R}^{24} = \tilde{f}_1(u)\tilde{P}_2^{11} + \tilde{f}_2(u)\tilde{P}_2^{22} + f(u)(P_2^{11} + P_2^{22} + iP_2^{12} - iP_2^{21}) \quad (2.19)$$

and

$$\check{R}^{2^4}(u) = h(u) \left(\sum_{i,j=1,2} \tilde{h}_{ij} \tilde{P}_2^{ij} + \left[P_2^{12} + P_2^{21} \right] \right) + f(u) (P_2^{11} + P_2^{22} + iP_2^{12} - iP_2^{21}). \quad (2.20)$$

(with arbitrary functions $\tilde{f}_{1,2}(u)$, $f(u)$ and $h(u)$ and arbitrary numbers \tilde{h}_{ij}) which satisfy to the YBE with 4×4 intertwiner matrix $R^{22}(u) = \mathbb{I}$. It means, that together with the transfer matrices with different spectral parameters, constructed via the given R -matrices, the monodromy matrices also are commuting. As there is no proper normalization for both matrices to give $\check{R}(u_0) = \mathbb{I}$ at some point u_0 , so we shall not try to investigate the chain models corresponding to such matrices.

2.2.2 The solutions $\check{R}^{4^4}(u)$.

According to (2.8) the decomposition $\otimes^2 \mathcal{I}_{\{3,1\}}^{(4)}$ contains four $\mathcal{I}_{\{3,1\}}^{(4)}$ -representations. One must note here, that although all $\mathcal{I}_{\{3,1\}}^{(4)}$ are isomorphic one to another, they have different sets of the eigenvalues of the k -operator. Schematically one can describe the representation $\mathcal{I}_{\{3,1\}}^{(4)} = \{v_+, v_0, v_-, u_0\}$ as follows

$$\begin{aligned} e \cdot \{v_+, v_0, v_-, u_0\} &= \{0, 0, v_0, v_+\}, \\ f \cdot \{v_+, v_0, v_-, u_0\} &= \{v_0, 0, 0, v_-\}, \\ k \cdot \{v_+, v_0, v_-, u_0\} &= \varepsilon \{v_+, -v_0, v_-, -u_0\}, \\ c \cdot \{v_+, v_0, v_-, u_0\} &= \{0, 0, 0, v_0\}. \end{aligned} \quad (2.21)$$

Some numerical coefficients' variation is possible in this schematic action, due to the normalization of the vectors. The sign $\varepsilon = \pm$ is positive for two representations and is negative for the other pair. This happens from the following reason. The fusion of the tensor product $V_2 \otimes V_2 \otimes V_2 \otimes V_2$ at general q is $V_5 \oplus \left[\bigoplus_{i=1}^3 V_3^i \right] \oplus \left[\bigoplus_{i=1}^2 V_1^i \right]$. At $q = i$ two three-dimensional and two one-dimensional representations deform into two indecomposable ones, $V_3 \oplus V_1 \Rightarrow \mathcal{I}_{\{3,1\}}^{(4)}$, with $\varepsilon = -$. Meanwhile the other two indecomposable representations emerge from the deformation and splitting to the direct sum in this way $V_5 \oplus V_3 \Rightarrow \mathcal{I}_{\{5,3\}}^{(8)} \Rightarrow \mathcal{I}_{\{3,1\}}^{(4)} \oplus \mathcal{I}_{\{3,1\}}^{(4)}$ (see the work [8] for details), with $\varepsilon = +$.

Let us denote four indecomposable representations by $\mathcal{I}_{\{3,1\}\pm}^{(4)i} = \{v_+, v_0, v_-, u_0\}_\pm^i$, $i = 1, 2$. The possible independent projectors are $P_{\mathcal{I}\varepsilon\eta}^{ij}$, $P'_{\mathcal{I}\varepsilon\eta}^{ij}$, where $\varepsilon, \eta \in \{+, -\}$ and $i, j \in \{1, 2\}$. The action of the projectors $P_{\mathcal{I}\varepsilon\varepsilon}^{ij}$, $P'_{\mathcal{I}\varepsilon\varepsilon}^{ij}$ corresponds to the description given in the previous sections,

$$P_{\mathcal{I}\varepsilon\varepsilon}^{ij} \cdot \{v_+, v_0, v_-, u_0\}_\varepsilon^j = \{v_+, v_0, v_-, u_0\}_\varepsilon^i, \quad (2.22)$$

$$P_{\mathcal{I}\varepsilon\varepsilon}^{ij} \cdot \{v_+, v_0, v_-, u_0\}_{\varepsilon}^j = \{0, 0, 0, v_0\}_{\varepsilon}^i. \quad (2.23)$$

Meanwhile, the action of the projectors $P_{\mathcal{I}\varepsilon\bar{\varepsilon}}^{ij}$, $P_{\mathcal{I}\bar{\varepsilon}\varepsilon}^{ij}$, where $\bar{\varepsilon}$ is the opposite sign of ε , can be defined in the following way,

$$P_{\mathcal{I}\varepsilon\bar{\varepsilon}}^{ij} \cdot \{v_+, v_0, v_-, u_0\}_{\bar{\varepsilon}}^j = \{v_0, 0, 0, v_-\}_{\varepsilon}^i, \quad (2.24)$$

$$P_{\mathcal{I}\bar{\varepsilon}\varepsilon}^{ij} \cdot \{v_+, v_0, v_-, u_0\}_{\varepsilon}^j = \{0, 0, v_0, v_+\}_{\bar{\varepsilon}}^i. \quad (2.25)$$

In summary there are 32 independent projectors or algebra invariants (in explicit form they are given in the Appendix) in the representation space $\bigotimes^4 V_2 = \bigotimes^2 \mathcal{I}_{\{3,1\}}^{(4)}$ and hence the R -matrix can be constructed by means of their sum with 32 coefficient functions (one of them can be chosen as 1 due to normalization freedom). At general q the number of the independent projectors is 14: P_5 , P_3^{ij} and P_1^{kr} with $i, j = 1, 2, 3$ and $k, r = 1, 2$.

The simplest solution at general q can be obtained just by the following tensor product on the vector space $V_2 \otimes V_2 \otimes V_2 \otimes V_2$, using the fundamental solution $\check{R}^{2,2}(u)$ on the spin- $\frac{1}{2}$ states (the descendant property has been used)

$$\check{R}^{4,4}(u) = \left(\mathbb{I} \otimes \check{R}^{2,2}(u) \otimes \mathbb{I} \right) \left(\check{R}^{2,2}(u) \otimes \mathbb{I} \otimes \mathbb{I} \right) \left(\mathbb{I} \otimes \mathbb{I} \otimes \check{R}^{2,2}(u) \right) \left(\mathbb{I} \otimes \check{R}^{2,2}(u) \otimes \mathbb{I} \right). \quad (2.26)$$

Here \mathbb{I} is the 2×2 unity operator defined on the space V^2 . This \check{R} -matrix can be expressed surely by the mentioned above 14 projectors. Some modifications are possible of this solution conditioned by the automorphisms of the algebra, but it does not change the nature of the solution. At the limit $q \rightarrow i$ the linear combination of the projectors P_5 , P_3^{ij} and P_1^{kr} in the $R^{4,4}$ can be expressed by the sum of the following fourteen projectors - $\left(P_{\mathcal{I}++}^{11} + P_{\mathcal{I}++}^{22} \right)$, $P_{\mathcal{I}--}^{11}$, $P_{\mathcal{I}--}^{22}$, $P_{\mathcal{I}--}^{12}$, $P_{\mathcal{I}--}^{21}$, $\left(P_{\mathcal{I}++}^{11} + P_{\mathcal{I}++}^{22} \right)$, $P_{\mathcal{I}--}^{11}$, $P_{\mathcal{I}--}^{22}$, $P_{\mathcal{I}--}^{12}$, $P_{\mathcal{I}--}^{21}$, $\left(P_{\mathcal{I}+-}^{11} - P_{\mathcal{I}+-}^{12} \right)$, $\left(P_{\mathcal{I}+-}^{21} - P_{\mathcal{I}+-}^{11} \right)$, $\left(P_{\mathcal{I}+-}^{21} - P_{\mathcal{I}+-}^{22} \right)$, $\left(P_{\mathcal{I}+-}^{22} - P_{\mathcal{I}+-}^{12} \right)$, which can be found as the limits $q \rightarrow i$ of the appropriate linear combinations of the projectors at general q . The explicit expression of $\check{R}^{4,4}(u)$ is the following (below $t = \tanh u$)

$$\begin{aligned} \check{R}^{4,4}(u) = & P_{\mathcal{I}++}^{11} + P_{\mathcal{I}++}^{22} + (1 - 2t^2 + t^3)P_{\mathcal{I}--}^{11} + (1 - 2t^2 - t^3)P_{\mathcal{I}--}^{22} + \\ & t(2 - t^2)[P_{\mathcal{I}--}^{12} - P_{\mathcal{I}--}^{21}] + it[P_{\mathcal{I}++}^{11} + P_{\mathcal{I}++}^{22}] + \frac{i}{2}t(-8 + t + 5t^2 - t^3)P_{\mathcal{I}--}^{11} + \\ & \frac{i}{2}t(4 - t - t^2 + t^3)P_{\mathcal{I}--}^{22} + \frac{i}{2}t(-6 - 3t + t^3)P_{\mathcal{I}--}^{12} + \frac{i}{2}t(-6 + 3t + 6t^2 - t^3)P_{\mathcal{I}--}^{21} + \\ & t(1 - t)\left(\frac{i}{2}[P_{\mathcal{I}+-}^{11} - P_{\mathcal{I}+-}^{12}] + [P_{\mathcal{I}+-}^{11} - P_{\mathcal{I}+-}^{21}]\right) + t(1 + t)\left(\frac{i}{2}[P_{\mathcal{I}+-}^{22} - P_{\mathcal{I}+-}^{21}] + [P_{\mathcal{I}+-}^{12} - P_{\mathcal{I}+-}^{22}]\right). \end{aligned} \quad (2.27)$$

From the previous example we can expect that at $q = i$ there will be a generalization of this matrix (as the matrix (2.15)) containing more than the mentioned 14 projectors, and having no analogue

at general q . It can be obtained by using one pair (containing only the generators e and $k^{\pm 1/2}$ or f and $k^{\pm 1/2}$) of Jimbo's composite relations (which now involve three equations for each of the generators e and f) derived for the case $R^{r'_1 \times r''_1 r'_2 \times r''_2}$. Simultaneous solution of all the relations will coincide exactly with (2.27).

A generalization of the solution (2.26) which exists at any q can be written as follows (now with dependence on three spectral parameters u, v, w , which leads to corresponding modifications in the spectral parameter dependence in the YBE (2.11))

$$\check{R}^{4,4}(u; v, w) = \left(\check{R}^{2,2}(v) \otimes \mathbb{I} \otimes \mathbb{I} \right) \left(\mathbb{I} \otimes \mathbb{I} \otimes \check{R}^{2,2}(w) \right) \left(\mathbb{I} \otimes \check{R}^{2,2}(u) \otimes \mathbb{I} \right) \left(\check{R}^{2,2}(u-v) \otimes \mathbb{I} \otimes \mathbb{I} \right) \left(\mathbb{I} \otimes \mathbb{I} \otimes \check{R}^{2,2}(u-w) \right) \left(\mathbb{I} \otimes \check{R}^{2,2}(u-v-w) \otimes \mathbb{I} \right). \quad (2.28)$$

The matrix (2.26) is the particular case of the expression (2.28) with the parameters $w = 0 = v$, note that $\check{R}^{2,2}(0) = \mathbb{I}$. The matrix representation of $\check{R}^{3,3}(u)$ in 4×4 -dimensional representation space equals to $\check{R}^{4,4}(u; 1, 1)$, as $\check{R}^{2,2}(1) = P_3$. This solution can be obtained also from Jimbo's ordinary relations (1.8). As in the previous case, this solution also admits adding to it some terms ($\approx P_1 \otimes P_1$) with arbitrary coefficient functions, vanishing after multiplication by $P_3 \otimes P_3$. The limit $q \rightarrow i$ can be evaluated as in the case of $R^{2,4(3)}(u)$. But we shall not explicitly consider it now, as well as the generalization of the solution (2.27), because we are interested in such (new) solutions which have the normalization property $\check{R}^{4,4}(u_0) = \mathbb{I}$ at some u_0 .

The increasing of the number of the independent projectors from 14 to 32 at $q = i$ allows to hope, that for the $\check{R}^{4,4}(u)$ -matrix besides of the solutions at general q there must be also new solutions to the YBE (2.11).

As we are interested in the solutions to the YBE at roots of unity, let us consider the $\check{R}^{4,4}$ -matrix in the form of the following linear expansion over all 32 projection operators

$$\check{R}^{4,4}(u) = \sum_{i,j,k=1}^2 \left(f_k^{ij}(u) P_{\mathcal{I} \varepsilon_k \varepsilon_k}^{ij} + f_k'^{ij}(u) P_{\mathcal{I} \varepsilon_k \varepsilon_k}^{\prime ij} + \bar{f}_k^{ij}(u) P_{\mathcal{I} \varepsilon_k \bar{\varepsilon}_k}^{ij} + \bar{f}_k'^{ij}(u) P_{\mathcal{I} \varepsilon_k \bar{\varepsilon}_k}^{\prime ij} \right). \quad (2.29)$$

Below we present a list of the spectral parameter dependent solutions for the particular cases (if the condition $\check{R}^{4,4}(0) = \mathbb{I}$ fulfills, then the full list for each case), when some functions in the expansion (2.29) are vanishing.

1. At the first let us look for a solution in the form of $\check{R}(u) = a\mathbb{I} + \sum_{ij \varepsilon} f_{\varepsilon}^{ij}(u) P_{\mathcal{I}\varepsilon\varepsilon}^{'ij}$. When $i = j$ we find one solution with few arbitrary parameters f_0^k ,

$$\check{R}(u) = \mathbb{I} + u(f_0^1 P_{\mathcal{I}--}^{'11} + f_0^2 P_{\mathcal{I}--}^{'22} + f_0^3 P_{\mathcal{I}++}^{'11} + f_0^3 P_{\mathcal{I}++}^{'22}). \quad (2.30)$$

When $f_0^1 = f_0^2 = f_0^3 = f_0$ then $\check{R}(u) = \mathbb{I} + u f_0 c^{2222}$, where c^{2222} is the representation of the Casimir operator c (1.5) on the space $V_2 \otimes V_2 \otimes V_2 \otimes V_2$. Note that the c -operator writes as a sum of the following four projectors: $P_{\mathcal{I}\varepsilon\varepsilon}^{'ii}$, $i = 1, 2$, $\varepsilon = \pm$, as the eigenvalues of the c -operator on the eigenvectors $\{v_+, v_0, v_-\}_{\varepsilon}^i$ are 0.

The solutions, when $i \neq j$ in the sum $\sum_{ij \varepsilon} f_{\varepsilon}^{ij}(u) P_{\mathcal{I}\varepsilon\varepsilon}^{'ij}$, are numerous. Here we are presenting almost the full list of them, some constant solutions may have been omitted (the numbers f_0, g_0, h_0, \dots and the functions $f(u), h(u), e(u)$ below are arbitrary, if there is no another notation)

$$\varepsilon = +$$

$$\check{R}(u) = \mathbb{I} + u(f_0 P_{\mathcal{I}++}^{'11} + g_0 P_{\mathcal{I}++}^{'22} + h_0 P_{\mathcal{I}++}^{'12} + e_0 P_{\mathcal{I}++}^{'21}), \quad (2.31)$$

$$\check{R}(u) = f(u) P_{\mathcal{I}++}^{'11} + g(u) P_{\mathcal{I}++}^{'22} + h(u) P_{\mathcal{I}++}^{'12} + e(u) P_{\mathcal{I}++}^{'21}. \quad (2.32)$$

As we can verify, the matrix (2.32) is not invertible and in the standard scheme of constructing commuting charges via the transfer matrices it is not usable. But the particular case of that matrix, namely,

$$\check{R}(u) = (g(u) + f_0 h(u)) P_{\mathcal{I}++}^{'11} + g(u) P_{\mathcal{I}++}^{'22} + h(u) P_{\mathcal{I}++}^{'12} + e_0 h(u) P_{\mathcal{I}++}^{'21}, \quad (2.33)$$

satisfies to $[\check{R}(u), \check{R}(w)] = 0$ and hence, the transfer matrices (as well as monodromy matrices) with different spectral parameters constructed by them are also commuting.

$$\varepsilon = -$$

$$\check{R}(u) = f(u) [P_{\mathcal{I}--}^{'11} + P_{\mathcal{I}--}^{'12} - P_{\mathcal{I}--}^{'22} - P_{\mathcal{I}--}^{'21}] + g(u) [P_{\mathcal{I}--}^{'12} + P_{\mathcal{I}--}^{'21} + f_0 (P_{\mathcal{I}--}^{'22} + P_{\mathcal{I}--}^{'21})], \quad (2.34)$$

$$\check{R}(u) = f(u) [P_{\mathcal{I}--}^{'11} - P_{\mathcal{I}--}^{'21}] + g(u) [P_{\mathcal{I}--}^{'12} - P_{\mathcal{I}--}^{'22}], \quad (2.35)$$

$$\check{R}(u) = f(u) [P_{\mathcal{I}--}^{'11} + P_{\mathcal{I}--}^{'12}] + g(u) [P_{\mathcal{I}--}^{'22} + P_{\mathcal{I}--}^{'21}]. \quad (2.36)$$

In the three equations above (2.34-2.36) the functions are not arbitrary, $\frac{f(u)}{g(u)} = u$ or $\frac{f(u)}{g(u)} = e^u$. The solutions with the property $\check{R}(0) = \mathbb{I}$ are the following

$$\check{R}(u) = \mathbb{I} + \frac{2(e^u - 1)}{(1 + e^u)(g_0^{1/2} - g_0^{-1/2})^2} \left[P_{\mathcal{I}--}^{11} + g_0 P_{\mathcal{I}--}^{12} - P_{\mathcal{I}--}^{22} - g_0^{-1} P_{\mathcal{I}--}^{21} \right], \quad (2.37)$$

$$\check{R}(u) = \mathbb{I} + u \left(g_0 [P_{\mathcal{I}--}^{11} + P_{\mathcal{I}--}^{12} - P_{\mathcal{I}--}^{22} - P_{\mathcal{I}--}^{21}] + h_0 [P_{\mathcal{I}--}^{11} + (1 - e_0) P_{\mathcal{I}--}^{12} + e_0 P_{\mathcal{I}--}^{22}] \right) \quad (2.38)$$

Among the constant solutions we separate the solution

$$\check{R} = c^{2\ 2\ 2\ 2} = \sum_{i, \varepsilon = \pm} P_{\mathcal{I}\varepsilon\varepsilon}^{ii},$$

note that at general q the Casimir operator $c^{2\ 2\ 2\ 2}$ does not satisfy to the YBE. Two another solutions,

$$\check{R} = P_{\mathcal{I}--}^{11} - P_{\mathcal{I}--}^{22} + P_{\mathcal{I}--}^{12} - P_{\mathcal{I}--}^{21} \quad \text{and} \quad \check{R} = \sum_i P_{\mathcal{I}++}^{ii}. \quad (2.39)$$

are connected with the solutions $\check{R}_{1,2}^{3\ 3}(u)$ taken in the limit $q \rightarrow i$ (after the multiplication by $(1 + q^2)^2$, i.e. the singular parts) written in the representation space $V_2 \otimes V_2 \otimes V_2 \otimes V_2$. The first one is the exact 16×16 -dimensional analogue of the mentioned matrices in the limit $q \rightarrow i$, the second one is obtained just by replacing the $c^{3\ 3}$ - and $\mathbb{I}^{3\ 3}$ -matrices by $c^{2\ 2\ 2\ 2}$ and $\mathbb{I}^{2\ 2\ 2\ 2}$ in the $\check{R}_{1,2}^{3\ 3}(u)$, which we can denote by $\check{R}_{1,2}^{2\ 2\ 2\ 2}(u)$ (it is not a solution at general q) and then taking the limit $q \rightarrow i$ (previously removing the singularities with multiplying by $(1 + q^2)^2$). There is an obvious connection between two matrices $P_{\mathcal{I}--}^{11} - P_{\mathcal{I}--}^{22} + P_{\mathcal{I}--}^{12} - P_{\mathcal{I}--}^{21} \approx \lim_{q \rightarrow i} \left((P_3 \otimes P_3) \check{R}_{1,2}^{2\ 2\ 2\ 2}(u) (P_3 \otimes P_3) \right)$.

2. As another class of the solutions we consider the matrices with the projectors $P_{\mathcal{I}\varepsilon\varepsilon}^{ij}$.

$$\begin{aligned} \check{R}(u) = & a\mathbb{I} + f^+(u)P_{\mathcal{I}++}^{11} + g^+(u)P_{\mathcal{I}++}^{22} + h^+(u)P_{\mathcal{I}++}^{12} + e^+(u)P_{\mathcal{I}++}^{21} \\ & + f^-(u)P_{\mathcal{I}--}^{11} + g^-(u)P_{\mathcal{I}--}^{22} + h^-(u)P_{\mathcal{I}--}^{12} + e^-(u)P_{\mathcal{I}--}^{21}. \end{aligned} \quad (2.40)$$

There are few constant solutions with such R -matrices. Putting $f^+(u) = g^+(u) = e^+(u) = h^+(u) = 0$ in (2.40) we find no solutions (constant or spectral parameter dependent) to the YBE. In contrast to this, when $f^-(u) = g^-(u) = e^-(u) = h^-(u) = 0$, there are numerous solutions, as presented below (2.41-2.46). Here we represent the spectral parameter dependent solutions (corresponding constant ones can be obtained as the limits $u \rightarrow \pm\infty$), for which $\check{R}(0) = \mathbb{I}$

$$\check{R}(u) = P_{\mathcal{I}++}^{11} + e^{2u} P_{\mathcal{I}++}^{22} + e^u (P_{\mathcal{I}--}^{11} + P_{\mathcal{I}--}^{22}). \quad (2.41)$$

$$\check{R}(u) = \mathbb{I} + (e^u - 1)P_{\mathcal{I}++}^{11}, \quad \check{R}(u) = \mathbb{I} + (e^u - 1)P_{\mathcal{I}++}^{22}, \quad (2.42)$$

$$\check{R}(u) = \mathbb{I} + (e^u - 1)P_{\mathcal{I}++}^{11} + (e^{-u} - 1)P_{\mathcal{I}++}^{22} + f_0(e^u - e^{-u})P_{\mathcal{I}++}^{12/21}. \quad (2.43)$$

We can continue the list of such solutions presenting a general solution with $a = 1$ and (f_0, g_0) are arbitrary)

$$\{f^+(u), g^+(u), e^+(u), h^+(u)\} = \frac{(e^u - 1)}{2\bar{f}_0} \{\pm g_0 + \bar{f}_0, \mp g_0 + \bar{f}_0, \mp 2f_0, \mp 2\}, \quad (2.44)$$

$$\bar{f}_0 = \sqrt{4f_0 + g_0^2}.$$

The solutions (2.42) as well as solutions like as (below ”/” means that all four possibilities are admissible)

$$\check{R}(u) = \mathbb{I} + (e^u - 1)P_{\mathcal{I}++}^{11/22} + e_0(e^u - 1)P_{\mathcal{I}++}^{12/21} \quad (2.45)$$

are the particular cases of the solution (2.44).

Besides of the listed solutions, there are simple rational solutions also

$$\check{R}(u) = \mathbb{I} + u P_{\mathcal{I}++}^{12/21}. \quad (2.46)$$

At the end of this subsection, we would like to mention, that our attempts to find the solutions with the matrices $\check{R}(u) = \mathbb{I} + f^\varepsilon(u)P_{\mathcal{I}\varepsilon\varepsilon}^{11} + g^\varepsilon(u)P_{\mathcal{I}\varepsilon\varepsilon}^{22} + h^\varepsilon(u)P_{\mathcal{I}\varepsilon\varepsilon}^{11} + e^\varepsilon(u)P_{\mathcal{I}\varepsilon\varepsilon}^{22}$, $\varepsilon = \pm$, where $h^+(u) \neq 0$ or $e^+(u) \neq 0$ for $\varepsilon = +$, bring us to the conclusion that there is no any solution to the YBE with such expansion.

3. Next we observe the solutions with the projectors $P_{\mathcal{I}\varepsilon\varepsilon}^{ij}$. Let $\check{R}(0) = \mathbb{I}$.

Here we obtain the following rational solutions

$$\begin{aligned} \check{R}(u) &= \mathbb{I} + u (f_0 P_{\mathcal{I}-+}^{11} + g_0 P_{\mathcal{I}-+}^{21} + e_0 P_{\mathcal{I}+-}^{21} + h_0 P_{\mathcal{I}+-}^{22}), \\ \check{R}(u) &= \mathbb{I} + u (f_0 (P_{\mathcal{I}+-}^{11} + P_{\mathcal{I}+-}^{12}) + e_0 (P_{\mathcal{I}+-}^{21} + P_{\mathcal{I}+-}^{22}) + g_0 (P_{\mathcal{I}-+}^{11} - P_{\mathcal{I}-+}^{21}) + h_0 (P_{\mathcal{I}-+}^{22} - P_{\mathcal{I}-+}^{12})), \\ \check{R}(u) &= \mathbb{I} + u (f_0 (P_{\mathcal{I}+-}^{11} + P_{\mathcal{I}+-}^{12}) + e_0 (P_{\mathcal{I}+-}^{21} + P_{\mathcal{I}+-}^{22}) + g_0 P_{\mathcal{I}-+}^{11} + h_0 P_{\mathcal{I}-+}^{21}), \quad (2.47) \\ \check{R}(u) &= \mathbb{I} + u (f_0 P_{\mathcal{I}+-}^{21} + e_0 P_{\mathcal{I}+-}^{22} + g_0 (P_{\mathcal{I}-+}^{11} - P_{\mathcal{I}-+}^{21}) + h_0 (P_{\mathcal{I}-+}^{22} - P_{\mathcal{I}-+}^{12})), \\ \check{R}(u) &= \mathbb{I} + u (f_0 (2i P_{\mathcal{I}+-}^{11} + 2i P_{\mathcal{I}+-}^{12} + P_{\mathcal{I}-+}^{12} - P_{\mathcal{I}-+}^{22}) + \\ &\quad (e_0 + 2ih_0 + 2ig_0) P_{\mathcal{I}+-}^{21} + e_0 P_{\mathcal{I}+-}^{22} + g_0 P_{\mathcal{I}-+}^{11} + h_0 P_{\mathcal{I}-+}^{21}) \end{aligned}$$

and trigonometric solutions

$$\begin{aligned} \check{R}(u) &= \mathbb{I} + \frac{1-e^u}{1+e^u} (\pm 2P_{\mathcal{I}+-}^{12} \mp iP_{\mathcal{I}-+}^{12} + f_0 (P_{\mathcal{I}-+}^{11} - 2iP_{\mathcal{I}+-}^{22}) + g_0 (P_{\mathcal{I}-+}^{21} + 2iP_{\mathcal{I}+-}^{21}) \\ &\quad + e_0 (P_{\mathcal{I}-+}^{22} - 2iP_{\mathcal{I}+-}^{11} - 2iP_{\mathcal{I}+-}^{12} - P_{\mathcal{I}-+}^{12})). \end{aligned} \quad (2.48)$$

Some solutions in (2.47) can coincide one with other for the particular choices of the arbitrary parameters f_0 , g_0 , e_0 and h_0 .

The solutions with the projectors $P_{\mathcal{I}\varepsilon\bar{\varepsilon}}^{ij}$ are quite similar to (2.47, 2.48).

$$\begin{aligned}
\check{R}(u) &= \mathbb{I} + u \left(f_0 P_{\mathcal{I}+-}^{11} + g_0 P_{\mathcal{I}+-}^{12} + e_0 P_{\mathcal{I}-+}^{12} + h_0 P_{\mathcal{I}-+}^{22} \right), \\
\check{R}(u) &= \mathbb{I} + u \left(f_0 (P_{\mathcal{I}+-}^{11} + P_{\mathcal{I}+-}^{12}) + g_0 (P_{\mathcal{I}+-}^{21} + P_{\mathcal{I}+-}^{22}) + e_0 (P_{\mathcal{I}-+}^{21} - P_{\mathcal{I}-+}^{11}) + h_0 (P_{\mathcal{I}-+}^{22} - P_{\mathcal{I}-+}^{12}) \right), \\
\check{R}(u) &= \mathbb{I} + u \left(f_0 (P_{\mathcal{I}+-}^{11} + P_{\mathcal{I}+-}^{12}) + e_0 (P_{\mathcal{I}+-}^{21} + P_{\mathcal{I}+-}^{22}) + g_0 P_{\mathcal{I}-+}^{12} + h_0 P_{\mathcal{I}-+}^{22} \right), \quad (2.49) \\
\check{R}(u) &= \mathbb{I} + u \left(f_0 P_{\mathcal{I}+-}^{12} + e_0 P_{\mathcal{I}+-}^{11} + g_0 (P_{\mathcal{I}-+}^{11} - P_{\mathcal{I}-+}^{21}) + h_0 (P_{\mathcal{I}-+}^{22} - P_{\mathcal{I}-+}^{12}) \right), \\
\check{R}(u) &= \mathbb{I} + u \left(f_0 (P_{\mathcal{I}-+}^{11} - P_{\mathcal{I}-+}^{21} + 2i P_{\mathcal{I}+-}^{21} + 2i P_{\mathcal{I}+-}^{22}) + \right. \\
&\quad \left. (e_0 + 2ih_0 + 2ig_0) P_{\mathcal{I}+-}^{11} + e_0 P_{\mathcal{I}+-}^{12} + g_0 P_{\mathcal{I}-+}^{22} + h_0 P_{\mathcal{I}-+}^{12} \right) \\
\check{R}(u) &= \mathbb{I} + \frac{1-e^u}{1+e^u} \left(\pm i P_{\mathcal{I}-+}^{21} \pm 2 P_{\mathcal{I}+-}^{21} + f_0 (2i P_{\mathcal{I}+-}^{11} + P_{\mathcal{I}-+}^{22}) + g_0 (P_{\mathcal{I}-+}^{12} - 2i P_{\mathcal{I}+-}^{12}) + \right. \\
&\quad \left. e_0 (2i P_{\mathcal{I}+-}^{22} + P_{\mathcal{I}-+}^{11} + 2i P_{\mathcal{I}+-}^{21} - P_{\mathcal{I}-+}^{21}) \right). \quad (2.50)
\end{aligned}$$

Of course, consideration of the other possible structures of the R -matrices with different combinations of the projector operators also will give new solutions.

Note. Here we do not display all the solutions $R^{44}(u)$ to the YBE at general q or at roots of unity. The full list of the solutions are obtained for some definite cases (grouped in the marked paragraphs 1-3, for the last two cases provided $\check{R}(0) = \mathbb{I}$). However the presented results at roots of unity demonstrate the existence of the solutions which cannot be obtained from the solutions at general q . The plain evidence of it is the presence in the solutions of the projectors $(P_{\mathcal{I}++}^{ij}, P_{\mathcal{I}\varepsilon\bar{\varepsilon}}^{ij})$, which (wholly or separately) do not coincide with the limit $q \rightarrow i$ of any linear combination of the projectors existing at general q . The peculiarities of the obtained solutions, i.e. their large number and variety (constant ones, solutions with rational, exponential or trigonometric dependence on the spectral parameter, solutions containing arbitrary functions), existence of the rich number of arbitrary parameters, argue the novelty of their nature.

3 Chain models corresponding to the solutions.

This section is devoted to the study of the integrable models which can be defined using the YBE solutions described above, via the transfer matrix approach [1, 21, 22].

Let us define quantum space of a chain with N sites as $\mathbf{A}_N = A_1 \otimes A_2 \cdots \otimes A_N$, where A_i is the vector space corresponding to the i -th site, and serves as a representation space of the algebra $sl_q(2)$. If to construct transfer matrix $\tau(u) = \text{tr}_a \prod_i R_{ai}(u)$, with the operators $R_{ai}(u)$ which act on the vector spaces $A_a \otimes A_i$, and coincide with the solutions to the YBE obtained at roots of unity, then the resulting quantum chain model with the Hamiltonian operator defined as the first logarithmic derivative of the transfer matrix at the normalization point u_0 ($\check{R}(u_0) = \mathbb{I}$) can be treated as an extended XXZ model at roots of unity. We intend to investigate the case when $q = i$, i.e. the case of the extended XX models.

We take $A_i = [\mathcal{I}_{\{3,1\}}^{(4)}]_i = [V_2]_{2i} \otimes [V_2]_{2i+1}$. The solution given by the expression (2.26) corresponds to the ordinary XX model, with the following lattice Hamiltonian ($k \equiv 2i - 1$)

$$\begin{aligned} H_{XX} &= J \sum_{k, \Delta k=2}^{2N} \left(\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + 2(\sigma_{k+1}^+ \sigma_{k+2}^- + \sigma_{k+1}^- \sigma_{k+2}^+) \right. \\ &\quad \left. + \sigma_{k+3}^+ \sigma_{k+4}^- + \sigma_{k+3}^- \sigma_{k+4}^+ + \frac{i}{2}(\sigma_k^z + \sigma_{k+1}^z - \sigma_{k+3}^z - \sigma_{k+4}^z) \right) \\ &= J \sum_{k, \Delta k=1}^{2N} \left(\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + \frac{i}{2}(\sigma_k^z - \sigma_{k+1}^z) \right). \end{aligned} \quad (3.1)$$

Here the cyclic boundary conditions $\sigma_1^k = \sigma_{2N+1}^k$ and $\sigma_2^k = \sigma_{2N+2}^k$ (with $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$) are imposed, and the terms with σ_i^z -operators, ensuring $sl_i(2)$ symmetry, are disappeared in the entire expression. The same Hamiltonian can be obtained, as it is well known, from the fundamental $R^{2,2}(u)$ -matrix at $q = i$. The appearing of the coupling constant J in (3.1) mathematically reflects the freedom of the scaling of the spectral parameter u . It must be real, in order to keep the hermicity of the Hamiltonian operator. But for the cases brought below, when the hermicity is broken, there is no general condition on J .

3.1 Extended XX models: non-Hermitian Hamiltonian operators.

Now let us write the Hamiltonian operators corresponding to the new obtained solutions. We shall observe few of them, so that to touch on all the obtained types of the solutions. We shall start with the construction of the model given by the R -matrix (2.30). The simplest case, which corresponds to the sum of the unity and Casimir operators, gives the following expression

$$H^c = \sum_{k, \Delta k=2}^{2N} \left(\sigma_k^+ \sigma_{k+3}^- + \sigma_k^- \sigma_{k+3}^+ + i\sigma_k^z(\sigma_{k+1}^+ \sigma_{k+3}^- + \sigma_{k+1}^- \sigma_{k+3}^+) - i(\sigma_k^+ \sigma_{k+2}^- + \sigma_k^- \sigma_{k+2}^+) \sigma_{k+3}^z \right)$$

$$\begin{aligned}
& +\sigma_k^z(\sigma_{k+1}^+\sigma_{k+2}^- + \sigma_{k+1}^-\sigma_{k+2}^+)\sigma_{k+3}^z - (\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+)\sigma_{k+2}^z\sigma_{k+3}^z - \sigma_k^z\sigma_{k+1}^z(\sigma_{k+2}^+\sigma_{k+3}^- + \sigma_{k+2}^-\sigma_{k+3}^+) \\
& + \frac{i}{2}(\sigma_k^z\sigma_{k+1}^z\sigma_{k+3}^z + \sigma_{k+1}^z\sigma_{k+2}^z\sigma_{k+3}^z - \sigma_k^z\sigma_{k+1}^z\sigma_{k+2}^z - \sigma_k^z\sigma_{k+2}^z\sigma_{k+3}^z). \tag{3.2}
\end{aligned}$$

And apparently, the Hamiltonian (3.2) in the representation of the scalar fermions, evaluated by means of the Jordan-Wigner transformations,

$$\sigma_i^+ = c_i \prod_{j=1}^{i-1} (1 - 2c_j^+ c_j), \quad \sigma_i^- = c_i^+ \prod_{j=1}^{i-1} (1 - 2c_j^+ c_j), \quad \sigma_i^z = 1 - 2c_i^+ c_i, \tag{3.3}$$

see as example [22, 18], contains interaction terms up to the sixth power of the fermion operators and, hence, is not free-fermionic as it was in the case (3.1). Also, it contains non-Hermitian terms. Note, that the next to nearest Hamiltonian derived from the fundamental $R^{2,2}(u)$ -matrix (i.e. second logarithmic derivative of the transfer matrix) contain terms like $\sigma_i^\pm \sigma_{i+1}^z \sigma_{i+2}^\mp$ ($= c_i^+ c_{i+2}$ or $c_{i+2}^+ c_i$), i.e. describes free fermions.

It is interesting to present the Hamiltonian operators corresponding to the new solutions (with the R -matrices which cannot be obtained as the limits at roots of unity of the matrices at general q). Such matrices are, as example, $\check{R}^{12/21}(u) = \mathcal{I} + uP_{++}^{12/21}$ (2.46). Hamiltonian operators corresponding to them are (in the spin and fermionic representations)

$$H_{++}^{12} = J \sum_{k, \Delta k=2}^{2N} \left(\sigma_{k+1}^+ \sigma_{k+2}^+ - i\sigma_k^+ \sigma_{k+1}^z \sigma_{k+2}^+ - \sigma_k^+ \sigma_{k+1}^+ \right) = \tag{3.4}$$

$$J \sum_i^N \left(\sigma_{2i}^+ \sigma_{2i+1}^+ - i\sigma_{2i-1}^+ \sigma_{2i}^z \sigma_{2i+1}^+ - \sigma_{2i-1}^+ \sigma_{2i}^+ \right) \Rightarrow J \sum_i^N \left(c_{2i+1} c_{2i} - i c_{2i+1} c_{2i-1} - c_{2i} c_{2i-1} \right),$$

$$H_{++}^{21} = J \sum_{k, \Delta k=2}^{2N} \left(\sigma_{k+1}^- \sigma_{k+2}^- - i\sigma_k^- \sigma_{k+1}^z \sigma_{k+2}^- - \sigma_k^- \sigma_{k+1}^- \right) = \tag{3.5}$$

$$J \sum_i^N \left(\sigma_{2i}^- \sigma_{2i+1}^- - i\sigma_{2i-1}^- \sigma_{2i}^z \sigma_{2i+1}^- - \sigma_{2i-1}^- \sigma_{2i}^- \right) \Rightarrow J \sum_i^N \left(c_{2i}^+ c_{2i+1}^+ - i c_{2i-1}^+ c_{2i+1}^+ - c_{2i-1}^+ c_{2i}^+ \right).$$

As we see they both are non-Hermitian free-fermionic operators.

Another Hamiltonian operators resulted from the new solutions, can be found from the matrices (2.30, 2.31, 2.37, 2.38, 2.40-2.48).

Among the mentioned solutions we can see that the matrix (2.43) at small u and at $f_0 = 0$ takes the form $\check{R}(u) = \mathbb{I} + u(P_{++}^{11} - P_{++}^{22})$, and hence the corresponding Hamiltonian writes as

$$H_{++} = J \sum_{k, \Delta k=2}^{2N} \left(i(\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ - \sigma_{k+1}^+ \sigma_{k+2}^- - \sigma_{k+1}^- \sigma_{k+2}^+) - \sigma_k^+ \sigma_{k+1}^z \sigma_{k+2}^- - \sigma_k^- \sigma_{k+1}^z \sigma_{k+2}^+ + \sigma_{k+1}^z \right) \tag{3.6}$$

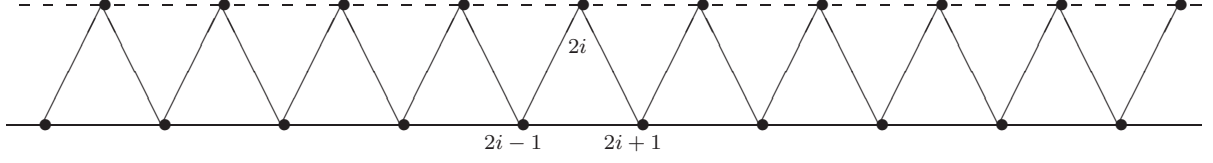


Figure 1: Graphical representations of the spin-chain Hamiltonians (3.4, 3.5, 3.7).

The corresponding fermionic representation of the Hamiltonian looks like as follows

$$H_{++}^f = J \sum_i^N \left(i(c_{2i-1}^+ c_{2i} + c_{2i}^+ c_{2i-1} - c_{2i}^+ c_{2i+1} - c_{2i+1}^+ c_{2i}) - c_{2i-1}^+ c_{2i+1} - c_{2i+1}^+ c_{2i-1} + 1 - 2c_{2i}^+ c_{2i} \right) \quad (3.7)$$

If in (2.43) $f_0 \neq 0$, then the additional term for the case of P_{I++}^{12} writes as $2f_0 J \sum_i^N (\sigma_{2i}^+ \sigma_{2i+1}^+ - \sigma_{2i-1}^+ \sigma_{2i}^+ - i\sigma_{2i-1}^+ \sigma_{2i}^z \sigma_{2i+1}^+)$ or, in the fermionic representation, $2f_0 J \sum_i^N (c_{2i+1} c_{2i} + c_{2i-1} c_{2i} + i c_{2i-1} c_{2i+1})$. For obtaining the case of P_{I++}^{21} the operators σ_i^+ and c_i one must change by the operators σ_i^- and c_i^+ .

In the graphical representation the Hamiltonian operators (3.4, 3.5, 3.7) can be depicted more apparently on the lattices, where the odd and even numbered spins are shown on two different chains. In Fig. 1 the spin (or fermionic) variables are attached on the sites noted by the dots. The next-to-nearest Hamiltonians (3.4, 3.5, 3.7) contain hopping terms only along the thick lines of the figure.

The particular solutions of (2.37) and (2.38),

$$\check{R}^\pm(u) = \mathbb{I} + u \left(P_{I--}^{\prime 11} - P_{I--}^{\prime 22} \pm (P_{I--}^{\prime 12} - P_{I--}^{\prime 21}) \right),$$

give rise to "factorized" Hamiltonian operators, which look like as

$$H_{--}^{factor+} = \sum_{k, \Delta k=2}^{2N} h_{k,k+1} h_{k+2,k+3} = \quad (3.8)$$

$$J^+ \sum_{k, \Delta k=2}^{2N} \left(\sigma_k^+ \sigma_{k+1}^- + \sigma_{k+1}^+ \sigma_k^- + \frac{i}{2} (\sigma_k^z - \sigma_{k+1}^z) \right) \left(\sigma_{k+2}^+ \sigma_{k+3}^- + \sigma_{k+3}^+ \sigma_{k+2}^- + \frac{i}{2} (\sigma_{k+2}^z - \sigma_{k+3}^z) \right),$$

$$H_{--}^{factor-} = \sum_{k, \Delta k=2}^{2N} h_{k,k+3} h_{k+1,k+2} = \quad (3.9)$$

$$J^- \sum_{k, \Delta k=2}^{2N} \left(\sigma_k^+ \sigma_{k+3}^- + \sigma_{k+3}^+ \sigma_k^- + \frac{i}{2} (\sigma_k^z - \sigma_{k+3}^z) \right) \left(\sigma_{k+1}^+ \sigma_{k+2}^- + \sigma_{k+2}^+ \sigma_{k+1}^- + \frac{i}{2} (\sigma_{k+1}^z - \sigma_{k+2}^z) \right).$$

Note, that the Hamiltonian of the ordinary XX model is $\sum_i^{2N} h_{i,i+1}$ and the second Hamiltonian (second logarithmic derivative of the transfer matrix) is proportional to $\sum_i^{2N} [h_{i,i+1}, h_{i+1,i+2}]$ [22].

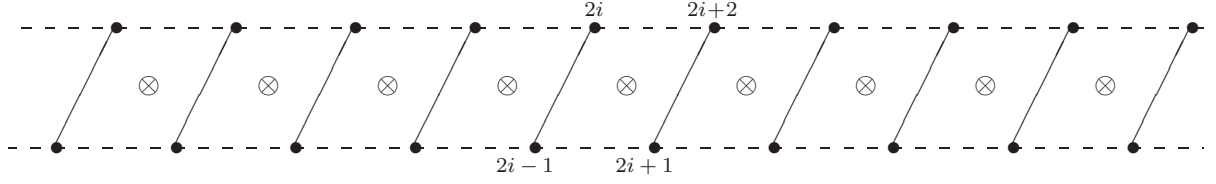


Figure 2: Graphical representations of the spin-chain Hamiltonian (3.8).

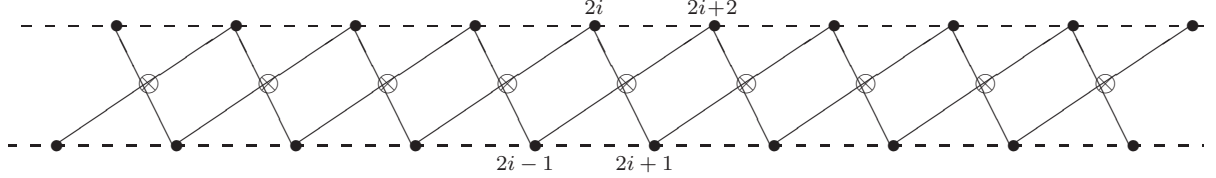


Figure 3: Graphical representations of the spin-chain Hamiltonian (3.9).

In the fermionic representation both of them contain only quadratic terms (describe free fermions), in the contrast of the Hamiltonian operators (3.8) and (3.9), which describe fermions with quartic interaction terms. Note also, that the term $h_{i,j} = \sigma_i^+ \sigma_j^- + \sigma_j^+ \sigma_i^- + \frac{i}{2}(\sigma_i^z - \sigma_j^z)$ is simply the Casimir operator c^2 defined on $[V_2]_i \otimes [V_2]_j$. And, particularly, the operator (3.8) can be represented also as $H_{--}^{factor+} = \sum_i^N h_{2i,2i+1} h_{2i+2,2i+3} = \sum_i^N [c^2]_i [c^2]_{i+1}$, being interpreted as a quadratic interaction between two nearest-neighbored four-dimensional indecomposable vector spaces.

In Figs. 2, 3 we represent the quartic Hamiltonians (3.8) and (3.9) in a graphical way: the local interactions take place between the spins (fermions) disposed on the four neighbored sites around the marked centers, with interaction terms presented by the products of two hopping terms h_{ij} along two thick lines, which are in the close vicinity of each center (Fig. 2) or are crossed in the centers (Fig. 3).

For completeness let us give also some Hamiltonian operators corresponding to the solutions (2.47-2.50). The second solution of (2.47) with the choice of the parameters $\{f_0, e_0, g_0, h_0\} = J_0\{1, 1, i/2, i/2\}$ leads to the following Hamiltonian

$$H_{+-} = J \sum_{k, \Delta k=2}^{2N} \left(\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + \frac{i}{2}(\sigma_k^z - \sigma_{k+1}^z) - \right. \\ \left. (\sigma_{k+1}^+ + i\sigma_{k+1}^- + (\sigma_k^- - i\sigma_k^+) \sigma_{k+1}^z)(\sigma_{k+2}^- + i\sigma_{k+2}^z \sigma_{k+3}^-) \right). \quad (3.10)$$

In the fermionic representation it is a non-Hermitian free fermionic operator

$$H_{+-}^f = J \sum_{i, \Delta i=2}^{2N} \left(c_k^+ c_{k+1} + c_{k+1}^+ c_k + i(c_{k+1}^+ c_{k+1} - c_k^+ c_k) - (c_k^+ + i c_{k+1}^+ - c_{k+1} + i c_k)(c_{k+2}^+ + i c_{k+3}^+) \right) \quad (3.11)$$

This Hamiltonian by its structure (as well as the operators (3.4) and (3.5)) resembles rather the Hamiltonian of the XY model.

A similar Hamiltonian operator we can find from the solutions (2.49), taking in the second matrix the following parameters $\{f_0, e_0, g_0, h_0\} = J'_0\{1, 1, i/2, -i/2\}$,

$$H'_{+-} = J \sum_{k, \Delta k=2}^{2N} \left(\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + \frac{i}{2} (\sigma_k^z - \sigma_{k+1}^z) - \right. \quad (3.12)$$

$$\left. (\sigma_{k+1}^- - i\sigma_{k+1}^+ - (\sigma_k^+ + i\sigma_k^-) \sigma_{k+1}^z) (\sigma_{k+2}^+ + i\sigma_{k+2}^z \sigma_{k+3}^+) \right).$$

The corresponding fermionic representation is

$$H'_{+-} = J \sum_{k, \Delta k=2}^{2N} \left(c_k^+ c_{k+1} + c_{k+1}^+ c_k + i(c_{k+1}^+ c_{k+1} - c_k^+ c_k) - (c_{k+1}^+ - ic_k^+ + c_k + ic_{k+1}) (c_{k+2} + ic_{k+3}) \right) \quad (3.13)$$

In the last examples given above we have dealt with the Hamiltonian functions which are homogeneous polynomials in respect of the fermionic operators (homogeneous polynomials of degree two (3.4, 3.5, 3.7, 3.11, 3.13)- containing only kinetic terms, or of degree four (3.8, 3.9)- only interaction terms). It is conditioned by our aim to choose more symmetric matrices among the YBE solutions. But of course, a large number of the solutions correspond to non-homogeneous Hamiltonians. The fermionic representation of the H in (3.2) contains terms with the second, fourth and sixth powers of the operators. As an illustration of the Hamiltonian with the four-fermionic interaction term together with a kinetic term, we can point the following Hamiltonian operators, corresponding to the simple solutions $\check{R}(u) = \mathbb{I} + uP_{\mathcal{I}-+}^{11}$, $\check{R}(u) = \mathbb{I} + uP_{\mathcal{I}-+}^{21}$ or $\check{R}(u) = \mathbb{I} + u(P_{\mathcal{I}-+}^{11} - P_{\mathcal{I}-+}^{21} + i(\Delta - 2)[P_{\mathcal{I}-+}^{21} + P_{\mathcal{I}-+}^{22}])$ (see (2.49)). For the last one the corresponding fermionic Hamiltonian is the following

$$H_{+-,\Delta}^f = J \sum_{i=1}^N \left(-2(c_{2i-1} + ic_{2i})(c_{2i+1} + ic_{2i+2}) + \right. \quad (3.14)$$

$$\left. \Delta [h_{2i-1,2i} c_{2i+1} c_{2i+2} + (ic_{2i-1}^+ c_{2i-1} c_{2i} + c_{2i-1} c_{2i}^+ c_{2i})(c_{2i+1} + ic_{2i+2})] \right).$$

Note. Taking into account that the local terms of the obtained new Hamiltonians connect two pairs of the neighboring spin- $\frac{1}{2}$ states (sometimes they restrict to three-spin interactions, as in (3.4, 3.5, 3.7)), reflecting the composite structure of the states on which the R -matrices are defined, one could relate the obtained models to those ones, being highly exploited in the strongly correlated systems, such as the dimer models, ladder (or zigzag) models. A general disadvantage which inheres

in the most of the discussed Hamiltonian operators is their non-hermicity. The quadratic in terms of the fermionic operators (i.e. free fermionic) Hamiltonian operators describe integrable models a priori, as the Fourier transformation allows to define the full eigen-system of such models. Hence, the Hermitian parts ($\frac{1}{2}[H + H^+]$, $\frac{1}{2i}[H - H^+]$) of a quadratic Hamiltonian also describe integrable models. But now they are fully diagonalizable and have real spectra, being in general with no $sl_i(2)$ symmetry (the Hamiltonian operator H^+ acquires the symmetry of the algebra $sl_{-i}(2)$, so the resulting Hamiltonian operators $\frac{1}{2}[H + H^+]$, $\frac{1}{2i}[H - H^+]$ are the combinations of the invariant operators in respect of $sl_i(2)$ and $sl_{-i}(2)$). As concerns the Hamiltonian operators with quartic and higher interactions, in each particular case there is need to check the integrability of the models defined by the Hermitian parts of the Hamiltonians.

And at the end of this section we would like to touch on the spectra of the discussed models with the free-fermionic behaviour. To obtain physically justified results and in order to deal with permissible transformations of the fermionic variables, we consider the Hermitian parts of the Hamiltonian operators. Particularly, for the fermionic H described in (3.7), in the Fourier basis of the chain discrete momenta,

$$c_{2i} = \frac{1}{\sqrt{N}} \sum_{p=1}^{2N} e^{-i\frac{\pi(2i)p}{N}} c_{1p}, \quad c_{2i+1} = \frac{1}{\sqrt{N}} \sum_{p=1}^{2N} e^{-i\frac{\pi(2i+1)p}{N}} c_{2p}, \quad (3.15)$$

the models with the Hamiltonian operators $\frac{1}{2}[H + H^+]$ and $\frac{1}{2i}[H - H^+]$, acquire the following energy spectra, correspondingly, $\{1, 2 \cos [\frac{2\pi p}{N}]\}$ and $\{\pm \sin [\pi \frac{p}{N}]\}$, $0 \leq p < N$. The Hermitian parts of the Hamiltonian operators (3.4), (3.5) have the eigenvalues, symmetric in respect of the origin. They are $\{\pm \cos [\pi \frac{p}{N}] \left(\sin [\pi \frac{p}{N}] \pm \sqrt{1 + \sin^2 [\pi \frac{p}{N}]} \right)\}$ and $\{\pm \cos [\pm \pi \frac{p}{N}]\}$ respectively, and here the eigenvectors are the combinations of the states with opposite momenta, c_{1p} , c_{2p} , $c_{1(N-p)}^+$, $c_{2(N-p)}^+$, $0 \leq p < N/2$ [18].

4 Treating of the indecomposable representations in the context of the dynamics of the systems. Non-unitary evolution operators.

In this section we want to observe the models with $sl_q(2)$ (as well as $osp(1|2)_q$) symmetry at roots of unity from another aspect. As we have seen the Hamiltonian operators which are constructed taking into account the indecomposable states are non-Hermitian. It means that the evolution

matrices of the corresponding models appear to be non-unitary. But in the recent decades there are numerous investigations of the systems with non-Hermitian Hamiltonians [19] and there is a chance that investigation of the new integrable models at roots of unity is not a pure mathematical analysis only.

The specific, peculiar character of the Hamiltonian operators at roots of unity consists of the presence of the indecomposable representations in the spectrum of the eigenstates. Let us observe the dynamics of such Hamiltonian systems. Suppose we have a chain with $2N$ sites with Hamiltonian e.g. (3.2). Let us consider the simplest case, when $N = 1$. The periodic boundary conditions imply $\sigma_3 = \sigma_1$, $\sigma_4 = \sigma_2$. After careful calculations we are coming to the following two-site Hamiltonian (with the normalized coefficient $J \rightarrow J/4$)

$$H = Jh_{1,2} = J\left(\sigma_1^+ \sigma_2^- + \sigma_2^+ \sigma_1^- + \frac{i}{2}(\sigma_1^z - \sigma_2^z)\right).$$

On the four-dimensional space $V_2 \otimes V_2$ this operator has the matrix form

$$H = J \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.1)$$

The vectors $|v_+\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|v_-\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $|v_0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}$

$\equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are the eigenstates of the Hamiltonian (4.1) with the

eigenvalue 0. Any state $|u_0\rangle = \frac{\gamma}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha |v_0\rangle$ with arbitrary α satisfies to the rela-

tion $H \cdot |u_0\rangle = J\gamma |v_0\rangle$. If to choose $|u_0\rangle = \frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}$ (with θ to be a real number), then the

scalar product defined as $(v^+, w) = (\langle v|)^* |w\rangle$ provides for the orthogonal and normalized vectors: $(v_\varepsilon^+, v_\eta) = \delta_{\varepsilon\eta}$, $(v_\varepsilon^+, u_0) = 0$, $(u_0^+, u_0) = 1$, where $\varepsilon, \eta = +, -, 0$. Note, that the ordinary scalar product $(v, w) = \langle v| |w\rangle$ (here and in the Appendix we have denoted by $\langle v|$ the transposed vector $(|v\rangle)^\tau$, without complex conjugation, in contrast to the usual convention, where $\langle v|$ means Hermitian conjugation) gives $(v_0, v_0) = 0$ (the vector with zero norm in the indecomposable representation).

In the quantum theory the definition (v^+, w) is used for measuring the probability of the system to occupy the given state.

Let us observe how the time evolution flows for the mentioned states. Usually considering the non-Hermitian models the authors try to avoid the problems coming with the non-unitary evolution matrices and the time-dependent norm [19, 20]. Let us see, what we shall have making a straightforward analysis. The solutions of the Shrödinger equation with the Hamiltonian (4.1) are the following time-dependent states: $|v_\varepsilon(t)\rangle = |v_\varepsilon\rangle$, $|u_0(t)\rangle = |u_0\rangle - itJ\gamma|v_0\rangle$. Note, that the norm of the state $|u_0(t)\rangle$ changes with time as follows $(u_0(t)^+, u_0(t)) = 1 + 4|Jt|^2$ (we use the vector $|u_0\rangle$ fixed above). Hence the normalized state

$$|\bar{u}_0(t)\rangle = \frac{|u_0(t)\rangle}{\sqrt{(u_0(t)^+, u_0(t))}} = \frac{|u_0\rangle + 2Je^{i\theta}t|v_0\rangle}{\sqrt{1 + 4|Jt|^2}}$$

in the limit $t \rightarrow \infty$ becomes $e^{i\theta} \frac{J}{|J|} |v_0\rangle$. We can conclude, that having an indecomposable representation $\{v_+, v_0, v_-, u_0\}$ at $t = 0$, the Hamiltonian operator (4.1) brings it at $t \rightarrow \infty$ to the representation space with actually three linearly independent vectors. Here in non-direct way we have put the function (role) of the evolution matrix $U(t) = e^{-itH}$ on the non-linear operator $\bar{U}(t)|u(0)\rangle = \frac{e^{-itH}|u(0)\rangle}{(u(0)^+ e^{itH^+}, e^{-itH} u(0))^{1/2}}$. This analysis easily can be extended for all the systems possessing the indecomposable states, which have not fully diagonalizable non-Hermitian Hamiltonian operators.

5 Summary

In this paper we have developed an approach to reveal all the possible solutions to the Yang-Baxter equations defined on the indecomposable representations. We have presented new integrable models with the symmetry $sl_q(2)$, when $q = i$. Like the ordinary XX model, these models also can be presented as one-dimensional chain models with the two-dimensional (spin-1/2) states at each site. The presented method can be extended for the another roots of q , as well as for the chains with other disposition and structure of the site's variables. The latter depends on the chosen indecomposable representations \mathcal{I}' and \mathcal{I}'' of the solutions $R_{\mathcal{I}'\mathcal{I}''}$ to the YBE. As an example at $q^3 = \pm 1$ (in this case the finite-dimensional non-reducible representations of the A -type are V_2 , V_3 , $\mathcal{I}_{\{4,2\}}^{(6)}$ and $\mathcal{I}_{\{5,1\}}^{(6)}$) we have tensor products $V_2 \otimes V_3 = \mathcal{I}_{\{4,2\}}^{(6)}$ and $\mathcal{I}_{\{4,2\}}^{(6)} \otimes \mathcal{I}_{\{4,2\}}^{(6)} = \left[\bigoplus^4 V_3 \right] \oplus \left[\bigoplus^2 \mathcal{I}_{\{5,1\}}^{(6)} \right] \oplus \left[\bigoplus^2 \mathcal{I}_{\{4,2\}}^{(6)} \right]$. It means, that having new solutions (which are not the descendants of the solutions at general q)

$R_{\mathcal{I}_1\mathcal{I}_2}$ with $\mathcal{I}_{1,2} = \mathcal{I}_{\{4,2\}}^{(6)}$ we can construct new models on a chain with the states at the sites defined as $A_i = [V_2]_{2i} \otimes [V_3]_{2i+1}$. The representation $\mathcal{I}_{\{5,1\}}^{(6)}$ emerges from the fusion $V_3 \otimes V_3 = \mathcal{I}_{\{5,1\}}^{(6)} \oplus V_1$, so the R -matrices defined on such representations can generate chain models with the local states being either $A_i = [\mathcal{I}_{\{5,1\}}^{(6)}]_i$ or $A_i = [V_3]_{2i} \otimes [V_3]_{2i+1}$.

Treatment of the representations, specific for the exceptional values of deformation parameter q , leads to the conclusion that we deal with pure "quantum"/deformed objects, which have no classical analogues. Some of the new solutions to the Yang-Baxter equations do not possess normalization property: have no regular point, where the R -matrix turns into unity operator. Other new solutions, which admit such point, do not satisfy the unitarity condition and the quantum chain Hamiltonian operators derived from such R -matrices are non-Hermitian. Another point is the drastic growth of the number of the solutions. As it is well-known at the exceptional values of q the center of the algebra is enlarged and new Casimir operators are appeared. Although the values of the operators of the extended center for the A -type representations do not give new characteristics, but the projection operators are closely related to the Casimir operators and the appearance of the large number of projectors reflects the extension of the symmetry of the system. Another manifestation of the same phenomena is the appearance of the rational (and exponential) solutions, which are not intrinsically inherited from the initially trigonometric solutions.

The large variety of the obtained Hamiltonians, only few of which were presented explicitly in the manuscript, needs more thorough and detailed analysis, which we intend to perform further.

Acknowledgments

The work is supported by the Armenian Grant No.11-1c028.

Appendix

Projection operators in case of degeneration of the Casimir operator's spectrum

If the coincidence of the eigenvalues of the Casimir operator c has a casual character and is not accompanied with the isomorphism of the representation spaces (which is possible, when q is a root of unity), then the set of the projection operators remains the same, and for determining them it

is enough to have an operator $c^{\frac{1}{n}}$ (or a well defined arbitrary $c_0 = \sum c_{0i} P^i$, where $c_{0i} \neq c_{0j}$), and to put it into (1.26) instead of c .

When the representations with the same eigenvalues of c are isomorphic, the situation changes. Inspection shows that in this case it is not possible to build all the projection operators by means of the polynomials in a single operator. The reason is, that along with the custom projection operators, here there are also operators P_r^{ij} which map the isomorphic spaces V_r^i, V_r^j with the same eigenvalues (c_r) of the Casimir operator, one to another (see Sections 1.2 and 1.3). Let us demonstrate it for the case, when

$$\mathcal{S} = V_r^1 \oplus V_r^2 \oplus \cdots \oplus V_r^n, \quad c = c_r \left(\sum_{i=1}^n P_r^i \right).$$

Then if one defines $\bar{c} = \sum_{ij} c_{ij} P_r^{ij}$, and tries to express the projectors P_r^{ij} as $\prod_k (a_k \bar{c} - h_k \mathbb{I})$, one can see, that it is not possible to define the identical projectors $P_r^i \equiv P_r^{ii}$, $\sum_i P_r^i = \mathbb{I}$, in this way, if $c_{ij} \neq 0$, $i \neq j$, neither the projectors P_r^{ij} can be defined. Using the properties of the projectors (1.20) one deduces $\prod_k^p (a_k \bar{c} - h_k \mathbb{I}) = \sum_{i,j}^n \mathcal{A}_{ij} P_r^{ij}$. For $n = 2$, we can see that, for any number p , we have $\mathcal{A}_{11} - \mathcal{A}_{22} = \mathcal{A}_{12}(c_{11} - c_{22})/c_{12} = \mathcal{A}_{21}(c_{11} - c_{22})/c_{21}$, so we cannot demand $\mathcal{A}_{ij} = \delta_{ik} \delta_{jr}$ for some k, r .

We need at least two operators, which commute with the algebra generators and have no degenerated eigen-spectrum. One can define the first one as $c^{\frac{1}{n}} = \sum_{i=1}^n c_r^i P_r^i$, taking not coinciding n roots c_r^i of c_r , $(c_r^i)^n = c_r$, and second one as $c_0 = \sum_{i \neq j} c_r^{ij} P_r^{ij}$ and one can demand $(c_0)^n = c$, too. By them we can construct

$$c^{\frac{1}{n}} = \sum_{i=1}^n c_r^i P_r^i, \quad c_0 = \sum_{i \neq j} c_r^{ij} P_r^{ij}, \quad (\text{A.1})$$

$$P_r^i = \prod_{k \neq i} \frac{c^{\frac{1}{n}} - c_r^k \mathbb{I}}{c_r^i - c_r^k}, \quad P_r^{ij} = P_r^i \frac{c_0}{c_r^{ij}} P_r^j. \quad (\text{A.2})$$

As well one can define two operators containing "upper/lower-diagonal" projectors P^{ii+1} (below the cyclic indexes i, j are defined by mod n):

$$c_{\pm}^{1/n} = \sum_i c_{ii\pm 1} P^{ii\pm 1}, \quad (c_{\pm}^{1/n})^n = c \Rightarrow \prod c_{ii\pm 1} = c_V, \quad (\text{A.3})$$

$$c_{\pm}^{1/n} c_{\mp}^{1/n} = \sum_i c_{ii\pm 1} c_{i\pm 1i} P^{ii}, \quad (\text{A.4})$$

$$P^{ii} = \prod_{k \neq i} \frac{c_{\pm}^{1/n} c_{\mp}^{1/n} - (c_{kk\pm 1} c_{k\pm 1k}) \mathbb{I}}{c_{ii\pm 1} c_{i\pm 1i} - c_{kk\pm 1} c_{k\pm 1k}}, \quad P^{ii\pm 1} = \frac{P^{ii} c_{\pm}^{1/n}}{c_{ii\pm 1}} = \frac{c_{\pm}^{1/n} P^{i\pm 1i\pm 1}}{c_{ii\pm 1}}, \quad (\text{A.5})$$

$$\text{if } i < j \quad P^{ij} = \overrightarrow{\prod}_{k=i}^{j-1} P^{kk+1}, \quad \text{if } i > j \quad P^{ij} = \overleftarrow{\prod}_{k=i}^{j+1} P^{kk-1}. \quad (\text{A.6})$$

Generalization for the cases when there are also isomorphic indecomposable representations with $c_{\mathcal{I}_i} = c_{\mathcal{I}_j}$ or $c_{\mathcal{I}_i} = c_{V_k}$, is straightforward. Suppose, we have $\mathcal{S} = \bigoplus_i^n V_r^i \oplus \bigoplus_k^p \mathcal{I}_k$, and

$$c = c_r \left(\sum_{i=1}^n P_r^i + \sum_{k=1}^p P_{\mathcal{I}_k} \right) + c'_{\mathcal{I}} \sum_{k=1}^p P'_{\mathcal{I}_k}.$$

Then let us define

$$c^{\frac{1}{n+p}} = \sum_{i=1}^n c_{r_i} P_r^i + \sum_{k=1}^p c_{\mathcal{I}_k} P_{\mathcal{I}_k} + \sum_{k=1}^p c'_{\mathcal{I}_k} P'_{\mathcal{I}_k},$$

so that $(c'_{\mathcal{I}_k})^{n+p} = c$, and hence $(c_{r_i})^{n+p} = (c_{\mathcal{I}_k})^{n+p} = c_r$, $c'_{\mathcal{I}_k} = \frac{c_{\mathcal{I}_k}}{(n+p)} \frac{c'_{\mathcal{I}}}{c_r}$ and the roots c_{r_i} , $c_{\mathcal{I}_k}$ do not coincide one with another. Obviously the projectors P_r^i , $P_{\mathcal{I}_k}$, $P'_{\mathcal{I}_k}$ can be constructed using the formulas (1.26), taking $c^{\frac{1}{n+p}}$ instead of c . Then we must define a second operator c_0 in order to determine the mixing projectors P_r^{ij} , $P_{\mathcal{I}}^{ij}$, $P'_{\mathcal{I}}^{ij}$. If the space V_r is isomorphic to the proper subspace U of \mathcal{I} , then there exist the following projectors too, $P_{\mathcal{I}V}^{ki}$ and $P'_{V\mathcal{I}}^{ik}$: $P_{\mathcal{I}V}^{ki} : V^i \Rightarrow U^k$, $P'_{V\mathcal{I}}^{ik} : \mathcal{U}^k \Rightarrow V^i$; on the other vectors they vanish. Here we supposed $\mathcal{I}^k = \mathcal{U}^k \cup \mathcal{U}'^k$, and $U^k \in \mathcal{U}^k$, $\dim[\mathcal{U}'^k] = \dim[U^k] = \dim[V^r]$.

$$c_0 = \sum_{i \neq j}^n c_r^{ij} P_r^{ij} + \sum_{i \neq j}^p (c_{\mathcal{I}}^{ij} P_{\mathcal{I}}^{ij} + c'_{\mathcal{I}}^{ij} P'_{\mathcal{I}}^{ij}) + \sum_{i=1}^n \sum_{k=1}^p (c_{\mathcal{I}V}^{ki} P_{\mathcal{I}V}^{ki} + c'_{V\mathcal{I}}^{ik} P'_{V\mathcal{I}}^{ik}).$$

The mixing projectors can be obtained by means of the ordinary ones and the operator c_0 as follows

$$P_r^{ij} = \frac{P_r^i c_0 P_r^j}{c_r^{ij}}, \quad P_{\mathcal{I}}^{ij} = \frac{P_{\mathcal{I}}^i c_0 P_{\mathcal{I}}^j}{c_{\mathcal{I}}^{ij}}, \quad P'_{\mathcal{I}}^{ij} = \frac{P_{\mathcal{I}}^i c_0}{c_{\mathcal{I}}^{ij}} \left(P_{\mathcal{I}}^j - \frac{c'_{\mathcal{I}}^{ij}}{c_{\mathcal{I}}^{ij}} P'_{\mathcal{I}}^j \right), \quad (\text{A.7})$$

$$P_{\mathcal{I}V}^{ki} = \frac{P_{\mathcal{I}}^k c_0 P_r^i}{c_{\mathcal{I}V}^{ki}}, \quad P'_{V\mathcal{I}}^{ik} = \frac{P_r^i c_0 P_{\mathcal{I}}^k}{c'_{V\mathcal{I}}^{ik}}. \quad (\text{A.8})$$

Projection operators at $q = i$: explicit form.

Choosing the vectors of the indecomposable representations so, that the action of the algebra generators look like as (2.21), the defining function for the existing 32 projection operators will be the following matrix

$$\mathcal{P}_{\mathcal{I}} = \sum_{i,j}^2 \sum_{\varepsilon, \eta} f_{\varepsilon\eta}^{ij} P_{\mathcal{I}\varepsilon\eta}^{ij} + \sum_{i,j}^2 \sum_{\varepsilon, \eta} f'_{\varepsilon\eta}^{ij} P'_{\mathcal{I}\varepsilon\eta}^{ij}, \quad (\text{A.9})$$

$$P_{\mathcal{I}\varepsilon\eta}^{ij} = \frac{d}{d f_{\varepsilon\eta}^{ij}} \mathcal{P}_{\mathcal{I}}, \quad P'^{ij}_{\mathcal{I}\varepsilon\eta} = \frac{d}{d f_{\varepsilon\eta}^{ij}} \mathcal{P}_{\mathcal{I}}. \quad (\text{A.10})$$

The projector operators are written by means of the states' vectors

$$\mathcal{I}_{\{3,1\}+}^{(4)1} = \{v_+, v_0, v_-, u_0\}_+^1 = \quad (\text{A.11})$$

$$\{\{1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}^\tau, \{0, -i, -1, 0, i, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0\}^\tau, \\ \{0, 0, 0, -1, 0, i, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0\}^\tau, \frac{1}{2}\{0, 1 - i, i - 1, 0, 1 + i, 0, 0, 0, 1 - i, 0, 0, 0, 0, 0, 0\}^\tau\},$$

$$\mathcal{I}_{\{3,1\}+}^{(4)2} = \{v_+, v_0, v_-, u_0\}_+^2 = \quad (\text{A.12})$$

$$\{\{0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -i, 0, -1, 0, 0, 0\}^\tau, \{0, 0, 0, 0, 0, 0, 0, -i, 0, 0, 0, -1, 0, i, 1, 0\}^\tau, \\ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}^\tau, \frac{1}{2}\{0, 0, 0, 0, 0, 0, 0, 1 - i, 0, 0, 0, -1 - i, 0, i - 1, 1 - i, 0\}^\tau\},$$

$$\mathcal{I}_{\{3,1\}-}^{(4)1} = \{v_+, v_0, v_-, u_0\}_-^1 = \quad (\text{A.13})$$

$$\{\{0, 0, 1, 0, -2i, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0\}^\tau, \{0, 0, 0, i, 0, 2, -i, 0, 0, -i, 0, 0, -i, 0, 0, 0\}^\tau, \\ \{0, 0, 0, 0, 0, 0, 0, -i, 0, 0, 0, 0, 0, -i, 0, 0\}^\tau, \frac{1}{2}\{0, 0, 0, 1, 0, i, 4, 0, 0, 2, -3i, 0, 1, 0, 0, 0\}^\tau\},$$

$$\mathcal{I}_{\{3,1\}-}^{(4)2} = \{v_+, v_0, v_-, u_0\}_-^2 = \quad (\text{A.14})$$

$$\{\{0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0\}^\tau, \{0, 0, 0, i, 0, 0, i, 0, 0, i, 2, 0, -i, 0, 0, 0\}^\tau, \\ \{0, 0, 0, 0, 0, 0, 0, i, 0, 0, 0, 2, 0, -i, 0, 0\}^\tau, \frac{1}{2}\{0, 0, 0, 4, 0, -3i, -1, 0, 0, 1, -i, 0, -2, 0, 0, 0\}^\tau\},$$

as follows (below, as usual, ket- and bra-vectors $|v\rangle$, $\langle v| = |v\rangle^\tau$ are corresponding to the vectors in column and row representations)

$$P_{\mathcal{I}\varepsilon\varepsilon}^{ij} = \sum_{k=+,-} \frac{i_\varepsilon |v_k\rangle \langle v_k|_\varepsilon^j}{\langle v_k|_\varepsilon^j \varepsilon | v_k\rangle} + \frac{i_\varepsilon |u_0\rangle \langle v_0|_\varepsilon^j}{\langle v_0|_\varepsilon^j \varepsilon | u_0\rangle} + \frac{1}{\langle u_0|_\varepsilon^j \varepsilon | v_0\rangle} \left(i_\varepsilon |v_0\rangle \langle u_0|_\varepsilon^j - \frac{\langle u_0|_\varepsilon^j \varepsilon | u_0\rangle}{\langle v_0|_\varepsilon^j \varepsilon | u_0\rangle} i_\varepsilon |v_0\rangle \langle v_0|_\varepsilon^j \right), \quad (\text{A.15})$$

$$P'^{ij}_{\mathcal{I}\varepsilon\varepsilon} = \frac{i_\varepsilon |v_0\rangle \langle v_0|_\varepsilon^j}{\langle v_0|_\varepsilon^j \varepsilon | u_0\rangle}, \quad (\text{A.16})$$

$$P_{\mathcal{I}\varepsilon\varepsilon}^{ij} = \frac{i_\varepsilon |v_0\rangle \langle v_+|_\varepsilon^j}{\langle v_+|_\varepsilon^j \varepsilon | v_+\rangle} + \frac{i_\varepsilon |v_- \rangle \langle v_0|_\varepsilon^j}{\langle v_0|_\varepsilon^j \varepsilon | u_0\rangle}, \quad P'^{ij}_{\mathcal{I}\varepsilon\varepsilon} = \frac{i_\varepsilon |v_0\rangle \langle v_-|_\varepsilon^j}{\langle v_-|_\varepsilon^j \varepsilon | v_- \rangle} + \frac{i_\varepsilon |v_+ \rangle \langle v_0|_\varepsilon^j}{\langle v_0|_\varepsilon^j \varepsilon | u_0\rangle}. \quad (\text{A.17})$$

There is an arbitrariness in the definition of the state vectors due to the normalization of the vectors, so all the vectors can be multiplied by some (non-zero) numbers, as well as, every vector $|u_0\rangle_\varepsilon^i$ can be shifted by $a_\varepsilon^i |v_0\rangle_\varepsilon^i$ with arbitrary number a_ε^i . The following transformations are possible: $|v'_k\rangle_\varepsilon^i = a_\varepsilon^i |v_k\rangle_\varepsilon^i$ (normalization), $|u'_0\rangle_\varepsilon^i = c_\varepsilon^i |u_0\rangle_\varepsilon^i + e_\varepsilon^i |v_0\rangle_\varepsilon^i$ (the behaviour of the u_0 -vectors), with arbitrary numbers a_ε^i , c_ε^i , e_ε^i . It explains the abundance of the arbitrary constants in the obtained YBE' solutions.

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